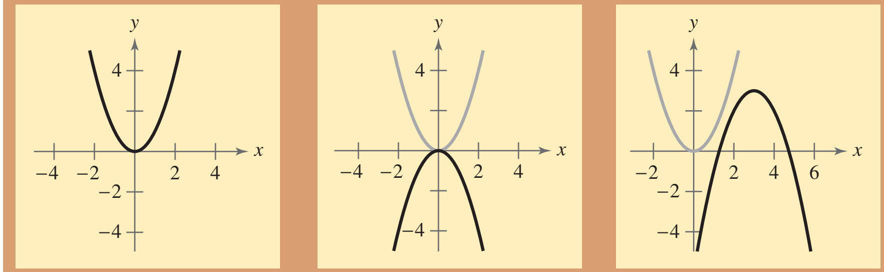
Chapter 1

Functions and Their Graphs

Selected Applications

Functions have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

- Rental Demand, Exercise 86, page 14
- Postal Regulations, Exercise 77, page 27
- Motor Vehicles, Exercise 83, page 28
- Fluid Flow, Exercise 92, page 40
- Finance, Exercise 58, page 50
- Bacteria, Exercise 81, page 61
- Consumer Awareness, Exercise 84, page 61
- Shoe Sizes, Exercises 103 and 104, page 71
- Cell Phones, Exercise 12, page 79



An equation in x and y defines a relationship between the two variables. The equation may be represented as a graph, providing another perspective on the relationship between x and y. In Chapter 1, you will learn how to write and graph linear equations, how to evaluate and find the domains and ranges of functions, and how to graph functions and their transformations.

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Refrigeration slows down the activity of bacteria in food so that it takes longer for the bacteria to spoil the food. The number of bacteria in a refrigerated food is a function of the amount of time the food has been out of refrigeration.

Introduction to Library of Parent Functions

In Chapter 1, you will be introduced to the concept of a *function*. As you proceed through the text, you will see that functions play a primary role in modeling real-life situations.

There are three basic types of functions that have proven to be the most important in modeling real-life situations. These functions are algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. These three types of functions are referred to as the *elementary functions*, though they are often placed in the two categories of *algebraic functions* and *transcendental functions*. Each time a new type of function is studied in detail in this text, it will be highlighted in a box similar to this one. The graphs of many of these functions are shown on the inside front cover of this text. A review of these functions can be found in the *Study Capsules*.

Algebraic Functions

These functions are formed by applying algebraic operations to the identity function $f(x) = x$.

Name	*Function*	*Location*
Linear	$f(x) = ax + b$	Section 1.1
Quadratic	$f(x) = ax^2 + bx + c$	Section 2.1
Cubic	$f(x) = ax^3 + bx^2 + cx + d$	Section 2.2
Polynomial	$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$	Section 2.2
Rational	$f(x) = \dfrac{N(x)}{D(x)}$, $N(x)$ and $D(x)$ are polynomial functions	Section 2.6
Radical	$f(x) = \sqrt[n]{P(x)}$	Section 1.2

Transcendental Functions

These functions cannot be formed from the identity function by using algebraic operations.

Name	*Function*	*Location*
Exponential	$f(x) = a^x, a > 0, a \neq 1$	Section 3.1
Logarithmic	$f(x) = \log_a x, x > 0, a > 0, a \neq 1$	Section 3.2
Trigonometric	$f(x) = \sin x, f(x) = \cos x, f(x) = \tan x,$	
	$f(x) = \csc x, f(x) = \sec x, f(x) = \cot x$	Section 4.4
Inverse Trigonometric	$f(x) = \arcsin x, f(x) = \arccos x, f(x) = \arctan x$	Section 4.7

Nonelementary Functions

Some useful nonelementary functions include the following.

Name	*Function*	*Location*
Absolute value	$f(x) = \lvert g(x)\rvert$, $g(x)$ is an elementary function	Section 1.2
Piecewise-defined	$f(x) = \begin{cases} 3x + 2, & x \geq 1 \\ -2x + 4, & x < 1 \end{cases}$	Section 1.2
Greatest integer	$f(x) = [\![g(x)]\!]$, $g(x)$ is an elementary function	Section 1.3
Data defined	Formula for temperature: $F = \frac{9}{5}C + 32$	Section 1.2

1.1 Lines in the Plane

What you should learn

- Find the slopes of lines.
- Write linear equations given points on lines and their slopes.
- Use slope-intercept forms of linear equations to sketch lines.
- Use slope to identify parallel and perpendicular lines.

Why you should learn it

The slope of a line can be used to solve real-life problems. For instance, in Exercise 87 on page 14, you will use a linear equation to model student enrollment at Penn State University.

Sky Bonillo/PhotoEdit

The Slope of a Line

In this section, you will study lines and their equations. The **slope** of a nonvertical line represents the number of units the line rises or falls vertically for each unit of horizontal change from left to right. For instance, consider the two points (x_1, y_1) and (x_2, y_2) on the line shown in Figure 1.1. As you move from left to right along this line, a change of $(y_2 - y_1)$ units in the vertical direction corresponds to a change of $(x_2 - x_1)$ units in the horizontal direction. That is,

$$y_2 - y_1 = \text{the change in } y$$

and

$$x_2 - x_1 = \text{the change in } x.$$

The slope of the line is given by the ratio of these two changes.

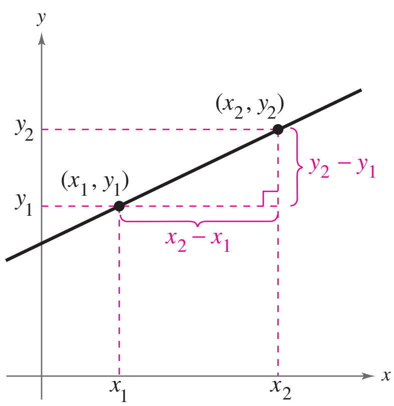


Figure 1.1

Definition of the Slope of a Line

The **slope** m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$

where $x_1 \neq x_2$.

When this formula for slope is used, the *order of subtraction* is important. Given two points on a line, you are free to label either one of them as (x_1, y_1) and the other as (x_2, y_2). However, once you have done this, you must form the numerator and denominator using the same order of subtraction.

$$\underbrace{m = \frac{y_2 - y_1}{x_2 - x_1}}_{\text{Correct}} \qquad \underbrace{m = \frac{y_1 - y_2}{x_1 - x_2}}_{\text{Correct}} \qquad \underset{\text{Incorrect}}{\cancel{m = \frac{y_2 - y_1}{x_1 - x_2}}}$$

Throughout this text, the term *line* always means a *straight* line.

Example 1 Finding the Slope of a Line

Find the slope of the line passing through each pair of points.

a. $(-2, 0)$ and $(3, 1)$ **b.** $(-1, 2)$ and $(2, 2)$ **c.** $(0, 4)$ and $(1, -1)$

Exploration

Use a graphing utility to compare the slopes of the lines $y = 0.5x$, $y = x$, $y = 2x$, and $y = 4x$. What do you observe about these lines? Compare the slopes of the lines $y = -0.5x$, $y = -x$, $y = -2x$, and $y = -4x$. What do you observe about these lines? (*Hint:* Use a *square setting* to guarantee a true geometric perspective.)

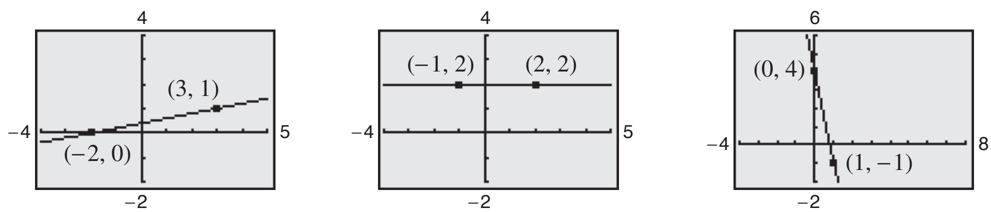
Solution

a. $$m = \frac{\overbrace{y_2 - y_1}^{\text{Difference in } y\text{-values}}}{\underbrace{x_2 - x_1}_{\text{Difference in } x\text{-values}}} = \frac{1 - 0}{3 - (-2)} = \frac{1}{3 + 2} = \frac{1}{5}$$

b. $$m = \frac{2 - 2}{2 - (-1)} = \frac{0}{3} = 0$$

c. $$m = \frac{-1 - 4}{1 - 0} = \frac{-5}{1} = -5$$

The graphs of the three lines are shown in Figure 1.2. Note that the *square setting* gives the correct "steepness" of the lines.



(a) **(b)** **(c)**

Figure 1.2

CHECKPOINT Now try Exercise 9.

The definition of slope does not apply to vertical lines. For instance, consider the points $(3, 4)$ and $(3, 1)$ on the vertical line shown in Figure 1.3. Applying the formula for slope, you obtain

$$m = \frac{4 - 1}{3 - 3} = \cancel{\frac{3}{0}}. \qquad \text{Undefined}$$

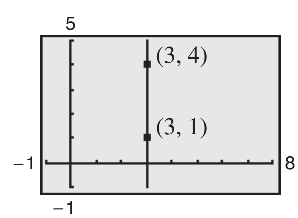


Figure 1.3

Because division by zero is undefined, the slope of a vertical line is undefined.

From the slopes of the lines shown in Figures 1.2 and 1.3, you can make the following generalizations about the slope of a line.

The Slope of a Line

1. A line with positive slope $(m > 0)$ *rises* from left to right.
2. A line with negative slope $(m < 0)$ *falls* from left to right.
3. A line with zero slope $(m = 0)$ is *horizontal*.
4. A line with undefined slope is *vertical*.

The Point-Slope Form of the Equation of a Line

If you know the slope of a line *and* you also know the coordinates of one point on the line, you can find an equation for the line. For instance, in Figure 1.4, let (x_1, y_1) be a point on the line whose slope is m. If (x, y) is any *other* point on the line, it follows that

$$\frac{y - y_1}{x - x_1} = m.$$

This equation in the variables x and y can be rewritten in the **point-slope form** of the equation of a line.

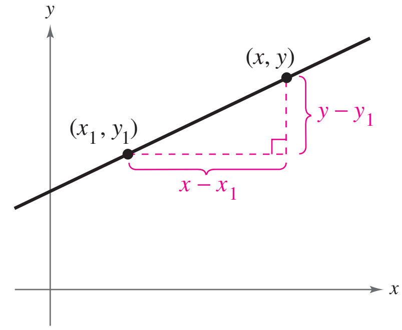


Figure 1.4

> **Point-Slope Form of the Equation of a Line**
>
> The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is
>
> $$y - y_1 = m(x - x_1).$$

The point-slope form is most useful for finding the equation of a line if you know at least one point that the line passes through and the slope of the line. You should remember this form of the equation of a line.

Example 2 The Point-Slope Form of the Equation of a Line

Find an equation of the line that passes through the point $(1, -2)$ and has a slope of 3.

Solution

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute for y_1, m, and x_1.
$y + 2 = 3x - 3$	Simplify.
$y = 3x - 5$	Solve for y.

The line is shown in Figure 1.5.

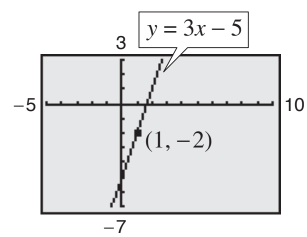


Figure 1.5

CHECKPOINT Now try Exercise 25.

The point-slope form can be used to find an equation of a nonvertical line passing through two points (x_1, y_1) and (x_2, y_2). First, find the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \; x_1 \neq x_2$$

Then use the point-slope form to obtain the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

This is sometimes called the **two-point form** of the equation of a line.

> **STUDY TIP**
>
> When you find an equation of the line that passes through two given points, you need to substitute the coordinates of only one of the points into the point-slope form. It does not matter which point you choose because both points will yield the same result.

Example 3 A Linear Model for Sales Prediction

During 2004, Nike's net sales were \$12.25 billion, and in 2005 net sales were \$13.74 billion. Write a linear equation giving the net sales y in terms of the year x. Then use the equation to predict the net sales for 2006. (Source: Nike, Inc.)

Solution

Let $x = 0$ represent 2000. In Figure 1.6, let $(4, 12.25)$ and $(5, 13.74)$ be two points on the line representing the net sales. The slope of this line is

$$m = \frac{13.74 - 12.25}{5 - 4} = 1.49. \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

By the point-slope form, the equation of the line is as follows.

$$y - 12.25 = 1.49(x - 4) \qquad \text{Write in point-slope form.}$$

$$y = 1.49x + 6.29 \qquad \text{Simplify.}$$

Now, using this equation, you can predict the 2006 net sales $(x = 6)$ to be

$$y = 1.49(6) + 6.29 = 8.94 + 6.29 = \$15.23 \text{ billion.}$$

CHECKPOINT Now try Exercise 45.

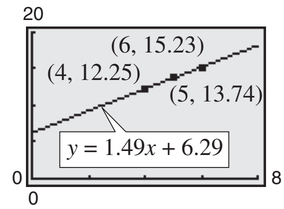


Figure 1.6

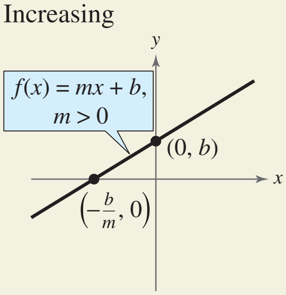
Library of Parent Functions: Linear Function

In the next section, you will be introduced to the precise meaning of the term *function*. The simplest type of function is a *linear function* of the form

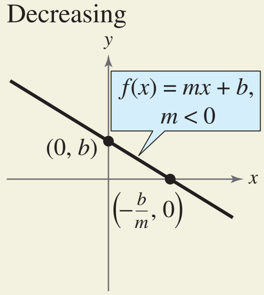
$$f(x) = mx + b.$$

As its name implies, the graph of a linear function is a line that has a slope of m and a y-intercept at $(0, b)$. The basic characteristics of a linear function are summarized below. (Note that some of the terms below will be defined later in the text.) A review of linear functions can be found in the *Study Capsules*.

Graph of $f(x) = mx + b, m > 0$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
x-intercept: $(-b/m, 0)$
y-intercept: $(0, b)$
Increasing



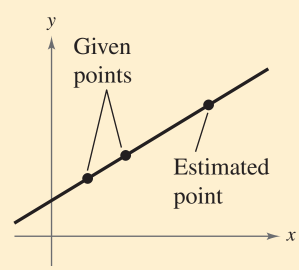
Graph of $f(x) = mx + b, m < 0$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
x-intercept: $(-b/m, 0)$
y-intercept: $(0, b)$
Decreasing



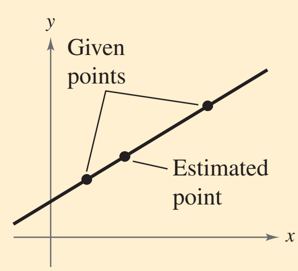
When $m = 0$, the function $f(x) = b$ is called a *constant function* and its graph is a horizontal line.

STUDY TIP

The prediction method illustrated in Example 3 is called **linear extrapolation.** Note in the top figure below that an extrapolated point does not lie between the given points. When the estimated point lies between two given points, as shown in the bottom figure, the procedure used to predict the point is called **linear interpolation.**



Linear Extrapolation



Linear Interpolation

Sketching Graphs of Lines

Many problems in coordinate geometry can be classified as follows.

1. Given a graph (or parts of it), find its equation.

2. Given an equation, sketch its graph.

For lines, the first problem is solved easily by using the point-slope form. This formula, however, is not particularly useful for solving the second type of problem. The form that is better suited to graphing linear equations is the **slope-intercept form** of the equation of a line, $y = mx + b$.

> **Slope-Intercept Form of the Equation of a Line**
>
> The graph of the equation
>
> $y = mx + b$
>
> is a line whose slope is m and whose y-intercept is $(0, b)$.

Example 4 Using the Slope-Intercept Form

Determine the slope and y-intercept of each linear equation. Then describe its graph.

a. $x + y = 2$ **b.** $y = 2$

Algebraic Solution

a. Begin by writing the equation in slope-intercept form.

$x + y = 2$ Write original equation.

$y = 2 - x$ Subtract x from each side.

$y = -x + 2$ Write in slope-intercept form.

From the slope-intercept form of the equation, the slope is -1 and the y-intercept is $(0, 2)$. Because the slope is negative, you know that the graph of the equation is a line that falls one unit for every unit it moves to the right.

b. By writing the equation $y = 2$ in slope-intercept form

$$y = (0)x + 2$$

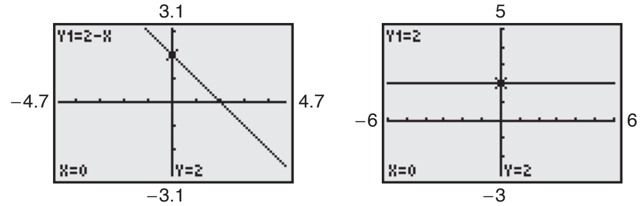
you can see that the slope is 0 and the y-intercept is $(0, 2)$. A zero slope implies that the line is horizontal.

CHECKPOINT Now try Exercise 47.

Graphical Solution

a. Solve the equation for y to obtain $y = 2 - x$. Enter this equation in your graphing utility. Use a decimal viewing window to graph the equation. To find the y-intercept, use the *value* or *trace* feature. When $x = 0$, $y = 2$, as shown in Figure 1.7(a). So, the y-intercept is $(0, 2)$. To find the slope, continue to use the *trace* feature. Move the cursor along the line until $x = 1$. At this point, $y = 1$. So the graph falls 1 unit for every unit it moves to the right, and the slope is -1.

b. Enter the equation $y = 2$ in your graphing utility and graph the equation. Use the *trace* feature to verify the y-intercept $(0, 2)$, as shown in Figure 1.7(b), and to see that the value of y is the same for all values of x. So, the slope of the horizontal line is 0.



(a) (b)

Figure 1.7

From the slope-intercept form of the equation of a line, you can see that a horizontal line ($m = 0$) has an equation of the form $y = b$. This is consistent with the fact that each point on a horizontal line through $(0, b)$ has a y-coordinate of b. Similarly, each point on a vertical line through $(a, 0)$ has an x-coordinate of a. So, a vertical line has an equation of the form $x = a$. This equation cannot be written in slope-intercept form because the slope of a vertical line is undefined. However, *every* line has an equation that can be written in the **general form**

$$Ax + By + C = 0 \qquad \text{General form of the equation of a line}$$

where A and B are not *both* zero.

Exploration

Graph the lines $y_1 = 2x + 1$, $y_2 = \frac{1}{2}x + 1$, and $y_3 = -2x + 1$ in the same viewing window. What do you observe?

Graph the lines $y_1 = 2x + 1$, $y_2 = 2x$, and $y_3 = 2x - 1$ in the same viewing window. What do you observe?

Summary of Equations of Lines

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$

Example 5 Different Viewing Windows

The graphs of the two lines

$$y = -x - 1 \qquad \text{and} \qquad y = -10x - 1$$

are shown in Figure 1.8. Even though the slopes of these lines are quite different (-1 and -10, respectively), the graphs seem misleadingly similar because the viewing windows are different.

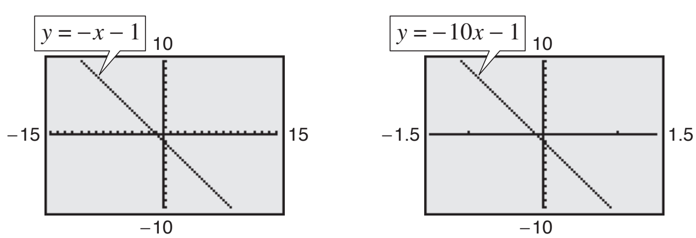
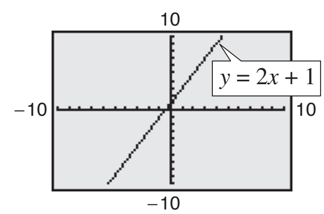


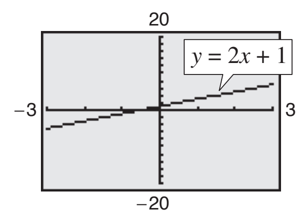
Figure 1.8

CHECKPOINT Now try Exercise 51.

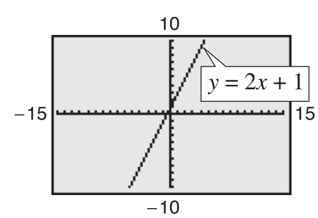
TECHNOLOGY TIP When a graphing utility is used to graph a line, it is important to realize that the graph of the line may not visually appear to have the slope indicated by its equation. This occurs because of the viewing window used for the graph. For instance, Figure 1.9 shows graphs of $y = 2x + 1$ produced on a graphing utility using three different viewing windows. Notice that the slopes in Figures 1.9(a) and (b) do not visually appear to be equal to 2. However, if you use a *square setting*, as in Figure 1.9(c), the slope visually appears to be 2.



(a)



(b)



(c)

Figure 1.9

Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular.

Parallel Lines

Two distinct nonvertical lines are **parallel** if and only if their slopes are equal. That is,

$$m_1 = m_2.$$

Example 6 Equations of Parallel Lines

Find the slope-intercept form of the equation of the line that passes through the point $(2, -1)$ and is parallel to the line $2x - 3y = 5$.

Solution

Begin by writing the equation of the given line in slope-intercept form.

$$2x - 3y = 5 \quad \text{Write original equation.}$$

$$-2x + 3y = -5 \quad \text{Multiply by } -1.$$

$$3y = 2x - 5 \quad \text{Add } 2x \text{ to each side.}$$

$$y = \frac{2}{3}x - \frac{5}{3} \quad \text{Write in slope-intercept form.}$$

Therefore, the given line has a slope of $m = \frac{2}{3}$. Any line parallel to the given line must also have a slope of $\frac{2}{3}$. So, the line through $(2, -1)$ has the following equation.

$$y - (-1) = \frac{2}{3}(x - 2) \quad \text{Write in point-slope form.}$$

$$y + 1 = \frac{2}{3}x - \frac{4}{3} \quad \text{Simplify.}$$

$$y = \frac{2}{3}x - \frac{7}{3} \quad \text{Write in slope-intercept form.}$$

Notice the similarity between the slope-intercept form of the original equation and the slope-intercept form of the parallel equation. The graphs of both equations are shown in Figure 1.10.

CHECKPOINT Now try Exercise 57(a).

Perpendicular Lines

Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is,

$$m_1 = -\frac{1}{m_2}.$$

TECHNOLOGY TIP

Be careful when you graph equations such as $y = \frac{2}{3}x - \frac{7}{3}$ with your graphing utility. A common mistake is to type in the equation as

Y1 = 2/3X − 7/3

which may not be interpreted by your graphing utility as the original equation. You should use one of the following formulas.

Y1 = 2X/3 − 7/3

Y1 = (2/3)X − 7/3

Do you see why?

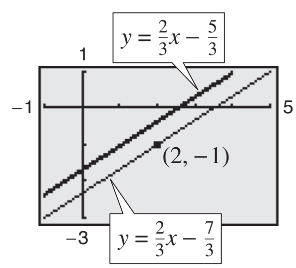


Figure 1.10

Example 7 Equations of Perpendicular Lines

Find the slope-intercept form of the equation of the line that passes through the point $(2, -1)$ and is perpendicular to the line

$$2x - 3y = 5.$$

Solution

From Example 6, you know that the equation can be written in the slope-intercept form $y = \frac{2}{3}x - \frac{5}{3}$. You can see that the line has a slope of $\frac{2}{3}$. So, any line perpendicular to this line must have a slope of $-\frac{3}{2}$ (because $-\frac{3}{2}$ is the negative reciprocal of $\frac{2}{3}$). So, the line through the point $(2, -1)$ has the following equation.

$$y - (-1) = -\tfrac{3}{2}(x - 2) \qquad \text{Write in point-slope form.}$$

$$y + 1 = -\tfrac{3}{2}x + 3 \qquad \text{Simplify.}$$

$$y = -\tfrac{3}{2}x + 2 \qquad \text{Write in slope-intercept form.}$$

The graphs of both equations are shown in Figure 1.11.

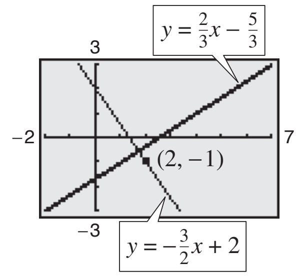


Figure 1.11

CHECKPOINT Now try Exercise 57(b).

Example 8 Graphs of Perpendicular Lines

Use a graphing utility to graph the lines

$$y = x + 1$$

and

$$y = -x + 3$$

in the same viewing window. The lines are supposed to be perpendicular (they have slopes of $m_1 = 1$ and $m_2 = -1$). Do they appear to be perpendicular on the display?

Solution

If the viewing window is nonsquare, as in Figure 1.12, the two lines will not appear perpendicular. If, however, the viewing window is square, as in Figure 1.13, the lines will appear perpendicular.

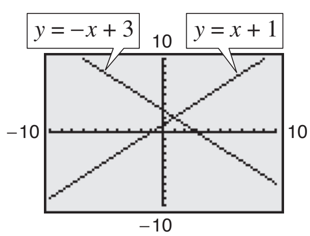


Figure 1.12

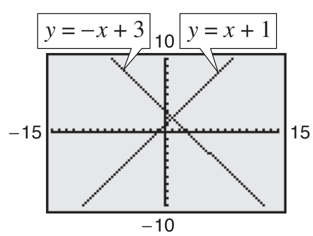


Figure 1.13

CHECKPOINT Now try Exercise 67.

1.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

1. Match each equation with its form.

(a) $Ax + By + C = 0$ (i) vertical line

(b) $x = a$ (ii) slope-intercept form

(c) $y = b$ (iii) general form

(d) $y = mx + b$ (iv) point-slope form

(e) $y - y_1 = m(x - x_1)$ (v) horizontal line

In Exercises 2–5, fill in the blanks.

2. For a line, the ratio of the change in y to the change in x is called the _______ of the line.

3. Two lines are _______ if and only if their slopes are equal.

4. Two lines are _______ if and only if their slopes are negative reciprocals of each other.

5. The prediction method _______ is the method used to estimate a point on a line that does not lie between the given points.

In Exercises 1 and 2, identify the line that has the indicated slope.

1. (a) $m = \frac{2}{3}$ (b) m is undefined. (c) $m = -2$

2. (a) $m = 0$ (b) $m = -\frac{3}{4}$ (c) $m = 1$

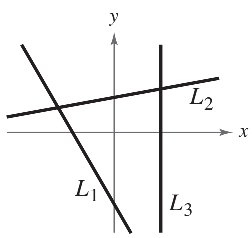


Figure for 1

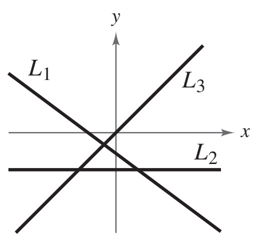


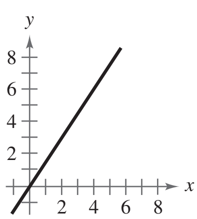
Figure for 2

In Exercises 3 and 4, sketch the lines through the point with the indicated slopes on the same set of coordinate axes.

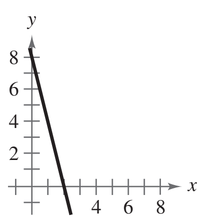
Point	*Slopes*			
3. $(2, 3)$	(a) 0	(b) 1	(c) 2	(d) -3
4. $(-4, 1)$	(a) 3	(b) -3	(c) $\frac{1}{2}$	(d) Undefined

In Exercises 5 and 6, estimate the slope of the line.

5.



6.



In Exercises 7–10, find the slope of the line passing through the pair of points. Then use a graphing utility to plot the points and use the *draw* feature to graph the line segment connecting the two points. (Use a *square setting*.)

7. $(0, -10), (-4, 0)$ **8.** $(2, 4), (4, -4)$

9. $(-6, -1), (-6, 4)$ **10.** $(-3, -2), (1, 6)$

In Exercises 11–18, use the point on the line and the slope of the line to find three additional points through which the line passes. (There are many correct answers.)

	Point	*Slope*
11.	$(2, 1)$	$m = 0$
12.	$(3, -2)$	$m = 0$
13.	$(1, 5)$	m is undefined.
14.	$(-4, 1)$	m is undefined.
15.	$(0, -9)$	$m = -2$
16.	$(-5, 4)$	$m = 2$
17.	$(7, -2)$	$m = \frac{1}{2}$
18.	$(-1, -6)$	$m = -\frac{1}{2}$

In Exercises 19–24, (a) find the slope and y-intercept (if possible) of the equation of the line algebraically, and (b) sketch the line by hand. Use a graphing utility to verify your answers to parts (a) and (b).

19. $5x - y + 3 = 0$ **20.** $2x + 3y - 9 = 0$

21. $5x - 2 = 0$ **22.** $3x + 7 = 0$

23. $3y + 5 = 0$ **24.** $-11 - 8y = 0$

In Exercises 25–32, find the general form of the equation of the line that passes through the given point and has the indicated slope. Sketch the line by hand. Use a graphing utility to verify your sketch, if possible.

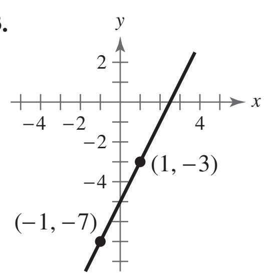
Point	*Slope*
25. $(0, -2)$	$m = 3$
26. $(-3, 6)$	$m = -2$
27. $(2, -3)$	$m = -\frac{1}{2}$
28. $(-2, -5)$	$m = \frac{3}{4}$
29. $(6, -1)$	m is undefined.
30. $(-10, 4)$	m is undefined.
31. $\left(-\frac{1}{2}, \frac{3}{2}\right)$	$m = 0$
32. $(2.3, -8.5)$	$m = 0$

In Exercises 33–42, find the slope-intercept form of the equation of the line that passes through the points. Use a graphing utility to graph the line.

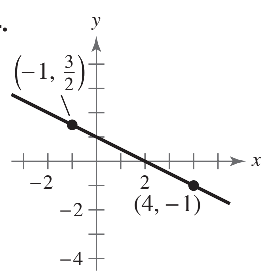
33. $(5, -1), (-5, 5)$
34. $(4, 3), (-4, -4)$
35. $(-8, 1), (-8, 7)$
36. $(-1, 4), (6, 4)$
37. $\left(2, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{4}\right)$
38. $(1, 1), \left(6, -\frac{2}{3}\right)$
39. $\left(-\frac{1}{10}, -\frac{3}{5}\right), \left(\frac{9}{10}, -\frac{9}{5}\right)$
40. $\left(\frac{3}{4}, \frac{3}{2}\right), \left(-\frac{4}{3}, \frac{7}{4}\right)$
41. $(1, 0.6), (-2, -0.6)$
42. $(-8, 0.6), (2, -2.4)$

In Exercises 43 and 44, find the slope-intercept form of the equation of the line shown.

43.



44.



45. ***Annual Salary*** A jeweler's salary was \$28,500 in 2004 and \$32,900 in 2006. The jeweler's salary follows a linear growth pattern. What will the jeweler's salary be in 2008?

46. ***Annual Salary*** A librarian's salary was \$25,000 in 2004 and \$27,500 in 2006. The librarian's salary follows a linear growth pattern. What will the librarian's salary be in 2008?

In Exercises 47–50, determine the slope and y-intercept of the linear equation. Then describe its graph.

47. $x - 2y = 4$
48. $3x + 4y = 1$
49. $x = -6$
50. $y = 12$

In Exercises 51 and 52, use a graphing utility to graph the equation using each of the suggested viewing windows. Describe the difference between the two graphs.

51. $y = 0.5x - 3$

Xmin = -5	Xmin = -2
Xmax = 10	Xmax = 10
Xscl = 1	Xscl = 1
Ymin = -1	Ymin = -4
Ymax = 10	Ymax = 1
Yscl = 1	Yscl = 1

52. $y = -8x + 5$

Xmin = -5	Xmin = -5
Xmax = 5	Xmax = 10
Xscl = 1	Xscl = 1
Ymin = -10	Ymin = -80
Ymax = 10	Ymax = 80
Yscl = 1	Yscl = 20

In Exercises 53–56, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.

53. L_1: $(0, -1), (5, 9)$
L_2: $(0, 3), (4, 1)$

54. L_1: $(-2, -1), (1, 5)$
L_2: $(1, 3), (5, -5)$

55. L_1: $(3, 6), (-6, 0)$
L_2: $(0, -1), \left(5, \frac{7}{3}\right)$

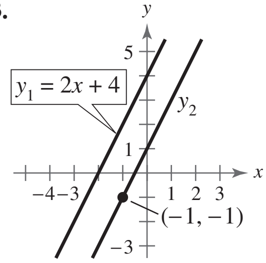
56. L_1: $(4, 8), (-4, 2)$
L_2: $(3, -5), \left(-1, \frac{1}{3}\right)$

In Exercises 57–62, write the slope-intercept forms of the equations of the lines through the given point (a) parallel to the given line and (b) perpendicular to the given line.

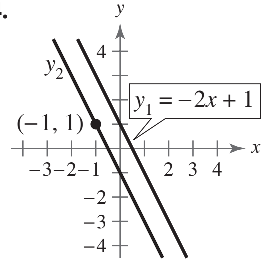
Point	*Line*
57. $(2, 1)$	$4x - 2y = 3$
58. $(-3, 2)$	$x + y = 7$
59. $\left(-\frac{2}{3}, \frac{7}{8}\right)$	$3x + 4y = 7$
60. $(-3.9, -1.4)$	$6x + 2y = 9$
61. $(3, -2)$	$x - 4 = 0$
62. $(-4, 1)$	$y + 2 = 0$

In Exercises 63 and 64, the lines are parallel. Find the slope-intercept form of the equation of line y_2.

63.

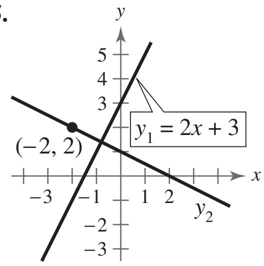


64.

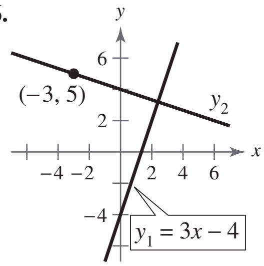


In Exercises 65 and 66, the lines are perpendicular. Find the slope-intercept form of the equation of line y_2.

65.



66.



Graphical Analysis **In Exercises 67–70, identify any relationships that exist among the lines, and then use a graphing utility to graph the three equations in the same viewing window. Adjust the viewing window so that each slope appears visually correct. Use the slopes of the lines to verify your results.**

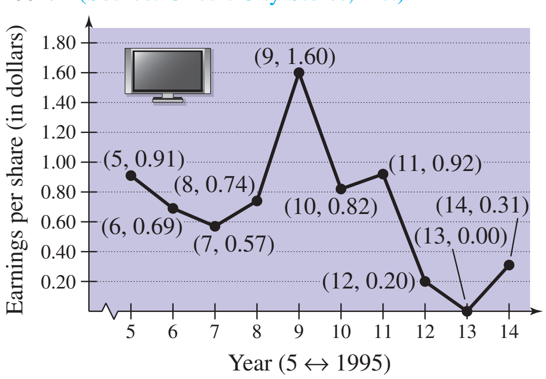
67. (a) $y = 2x$ (b) $y = -2x$ (c) $y = \frac{1}{2}x$

68. (a) $y = \frac{2}{3}x$ (b) $y = -\frac{3}{2}x$ (c) $y = \frac{2}{3}x + 2$

69. (a) $y = -\frac{1}{2}x$ (b) $y = -\frac{1}{2}x + 3$ (c) $y = 2x - 4$

70. (a) $y = x - 8$ (b) $y = x + 1$ (c) $y = -x + 3$

71. ***Earnings per Share*** The graph shows the earnings per share of stock for Circuit City for the years 1995 through 2004. (Source: Circuit City Stores, Inc.)



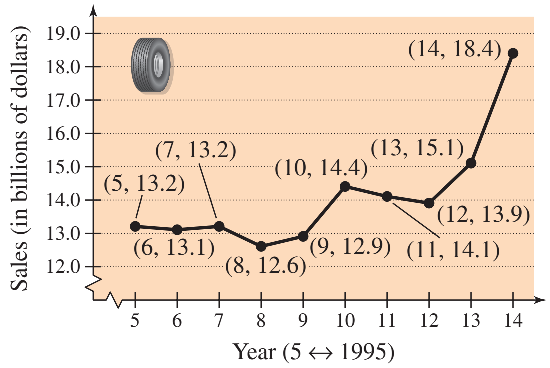
(a) Use the slopes to determine the years in which the earnings per share of stock showed the greatest increase and greatest decrease.

(b) Find the equation of the line between the years 1995 and 2004.

(c) Interpret the meaning of the slope of the equation from part (b) in the context of the problem.

(d) Use the equation from part (b) to estimate the earnings per share of stock in the year 2010. Do you think this is an accurate estimation? Explain.

72. ***Sales*** The graph shows the sales (in billions of dollars) for Goodyear Tire for the years 1995 through 2004, where $t = 5$ represents 1995. (Source: Goodyear Tire)



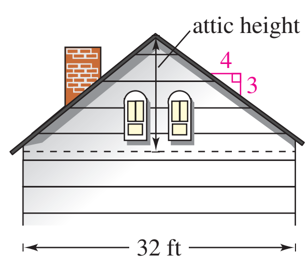
(a) Use the slopes to determine the years in which the sales for Goodyear Tire showed the greatest increase and the smallest increase.

(b) Find the equation of the line between the years 1995 and 2004.

(c) Interpret the meaning of the slope of the equation from part (b) in the context of the problem.

(d) Use the equation from part (b) to estimate the sales for Goodyear Tire in the year 2010. Do you think this is an accurate estimation? Explain.

73. ***Height*** The "rise to run" ratio of the roof of a house determines the steepness of the roof. The rise to run ratio of the roof in the figure is 3 to 4. Determine the maximum height in the attic of the house if the house is 32 feet wide.

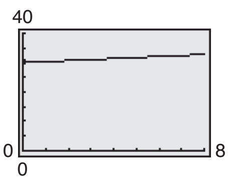


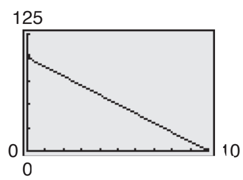
74. ***Road Grade*** When driving down a mountain road, you notice warning signs indicating that it is a "12% grade." This means that the slope of the road is $-\frac{12}{100}$. Approximate the amount of horizontal change in your position if you note from elevation markers that you have descended 2000 feet vertically.

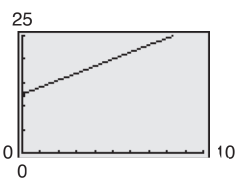
Rate of Change **In Exercises 75–78, you are given the dollar value of a product in 2006 and the rate at which the value of the product is expected to change during the next 5 years. Write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 6$ represent 2006.)**

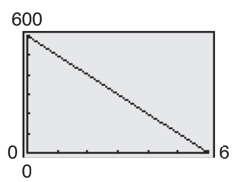
	2006 Value	*Rate*
75.	$2540	$125 increase per year
76.	$156	$4.50 increase per year
77.	$20,400	$2000 decrease per year
78.	$245,000	$5600 decrease per year

Graphical Interpretation **In Exercises 79–82, match the description with its graph. Determine the slope of each graph and how it is interpreted in the given context. [The graphs are labeled (a), (b), (c), and (d).]**

(a)


(b)


(c)


(d)


79. You are paying $10 per week to repay a $100 loan.

80. An employee is paid $12.50 per hour plus $1.50 for each unit produced per hour.

81. A sales representative receives $30 per day for food plus $.35 for each mile traveled.

82. A computer that was purchased for $600 depreciates $100 per year.

83. ***Depreciation*** A school district purchases a high-volume printer, copier, and scanner for $25,000. After 10 years, the equipment will have to be replaced. Its value at that time is expected to be $2000.

(a) Write a linear equation giving the value V of the equipment during the 10 years it will be used.

(b) Use a graphing utility to graph the linear equation representing the depreciation of the equipment, and use the *value* or *trace* feature to complete the table.

t	0	1	2	3	4	5	6	7	8	9	10
V											

(c) Verify your answers in part (b) algebraically by using the equation you found in part (a).

84. ***Meteorology*** Recall that water freezes at 0°C (32°F) and boils at 100°C (212°F).

(a) Find an equation of the line that shows the relationship between the temperature in degrees Celsius C and degrees Fahrenheit F.

(b) Use the result of part (a) to complete the table.

C		$-10°$	10°			177°
F	0°			68°	90°	

85. ***Cost, Revenue, and Profit*** A contractor purchases a bulldozer for $36,500. The bulldozer requires an average expenditure of $5.25 per hour for fuel and maintenance, and the operator is paid $11.50 per hour.

(a) Write a linear equation giving the total cost C of operating the bulldozer for t hours. (Include the purchase cost of the bulldozer.)

(b) Assuming that customers are charged $27 per hour of bulldozer use, write an equation for the revenue R derived from t hours of use.

(c) Use the profit formula $(P = R - C)$ to write an equation for the profit derived from t hours of use.

(d) Use the result of part (c) to find the break-even point (the number of hours the bulldozer must be used to yield a profit of 0 dollars).

86. ***Rental Demand*** A real estate office handles an apartment complex with 50 units. When the rent per unit is $580 per month, all 50 units are occupied. However, when the rent is $625 per month, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent p and the demand x is linear.

(a) Write the equation of the line giving the demand x in terms of the rent p.

(b) Use a graphing utility to graph the demand equation and use the *trace* feature to estimate the number of units occupied when the rent is $655. Verify your answer algebraically.

(c) Use the demand equation to predict the number of units occupied when the rent is lowered to $595. Verify your answer graphically.

87. ***Education*** In 1991, Penn State University had an enrollment of 75,349 students. By 2005, the enrollment had increased to 80,124. (Source: Penn State Fact Book)

(a) What was the average annual change in enrollment from 1991 to 2005?

(b) Use the average annual change in enrollment to estimate the enrollments in 1984, 1997, and 2000.

(c) Write the equation of a line that represents the given data. What is its slope? Interpret the slope in the context of the problem.

88. ***Writing*** Using the results of Exercise 87, write a short paragraph discussing the concepts of *slope* and *average rate of change*.

Synthesis

True or False? **In Exercises 89 and 90, determine whether the statement is true or false. Justify your answer.**

89. The line through $(-8, 2)$ and $(-1, 4)$ and the line through $(0, -4)$ and $(-7, 7)$ are parallel.

90. If the points $(10, -3)$ and $(2, -9)$ lie on the same line, then the point $\left(-12, -\frac{37}{2}\right)$ also lies on that line.

Exploration **In Exercises 91–94, use a graphing utility to graph the equation of the line in the form**

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, b \neq 0.$$

Use the graphs to make a conjecture about what a and b represent. Verify your conjecture.

91. $\frac{x}{5} + \frac{y}{-3} = 1$ **92.** $\frac{x}{-6} + \frac{y}{2} = 1$

93. $\frac{x}{4} + \frac{y}{-\frac{2}{3}} = 1$ **94.** $\frac{x}{\frac{1}{2}} + \frac{y}{5} = 1$

In Exercises 95–98, use the results of Exercises 91–94 to write an equation of the line that passes through the points.

95. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

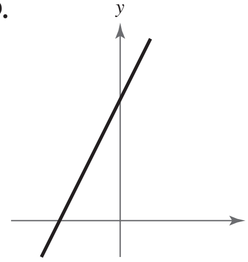
96. x-intercept: $(-5, 0)$
y-intercept: $(0, -4)$

97. x-intercept: $\left(-\frac{1}{6}, 0\right)$
y-intercept: $\left(0, -\frac{2}{3}\right)$

98. x-intercept: $\left(\frac{3}{4}, 0\right)$
y-intercept: $\left(0, \frac{4}{5}\right)$

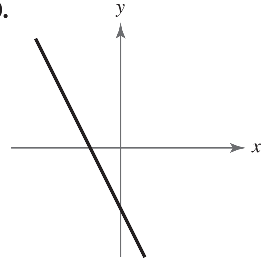
Library of Parent Functions **In Exercises 99 and 100, determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)**

99.



(a) $2x - y = -10$
(b) $2x + y = 10$
(c) $x - 2y = 10$
(d) $x + 2y = 10$

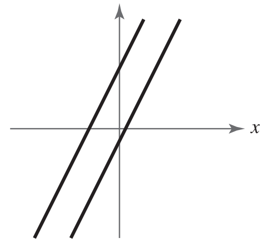
100.



(a) $2x + y = 5$
(b) $2x + y = -5$
(c) $x - 2y = 5$
(d) $x - 2y = -5$

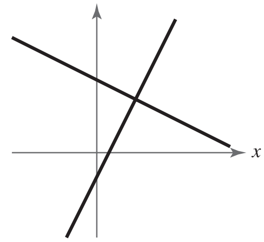
Library of Parent Functions **In Exercises 101 and 102, determine which pair of equations may be represented by the graphs shown.**

101.



(a) $2x - y = 5$
$2x - y = 1$
(b) $2x + y = -5$
$2x + y = 1$
(c) $2x - y = -5$
$2x - y = 1$
(d) $x - 2y = -5$
$x - 2y = -1$

102.



(a) $2x - y = 2$
$x + 2y = 12$
(b) $x - y = 1$
$x + y = 6$
(c) $2x + y = 2$
$x - 2y = 12$
(d) $x - 2y = 2$
$x + 2y = 12$

103. ***Think About It*** Does every line have both an x-intercept and a y-intercept? Explain.

104. ***Think About It*** Can every line be written in slope-intercept form? Explain.

105. ***Think About It*** Does every line have an infinite number of lines that are parallel to the given line? Explain.

106. ***Think About It*** Does every line have an infinite number of lines that are perpendicular to the given line? Explain.

Skills Review

In Exercises 107–112, determine whether the expression is a polynomial. If it is, write the polynomial in standard form.

107. $x + 20$ **108.** $3x - 10x^2 + 1$

109. $4x^2 + x^{-1} - 3$ **110.** $2x^2 - 2x^4 - x^3 + 2$

111. $\frac{x^2 + 3x + 4}{x^2 - 9}$ **112.** $\sqrt{x^2 + 7x + 6}$

In Exercises 113–116, factor the trinomial.

113. $x^2 - 6x - 27$ **114.** $x^2 - 11x + 28$

115. $2x^2 + 11x - 40$ **116.** $3x^2 - 16x + 5$

117. ***Make a Decision*** To work an extended application analyzing the numbers of bachelor's degrees earned by women in the United States from 1985 to 2005, visit this textbook's *Online Study Center.* (Data Source: U.S. Census Bureau)

The *Make a Decision* exercise indicates a multipart exercise using large data sets. Go to this textbook's *Online Study Center* to view these exercises.

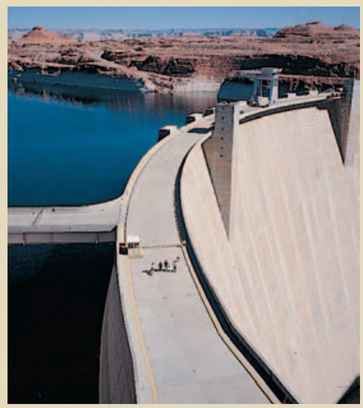
1.2 Functions

What you should learn

- Decide whether a relation between two variables represents a function.
- Use function notation and evaluate functions.
- Find the domains of functions.
- Use functions to model and solve real-life problems.
- Evaluate difference quotients.

Why you should learn it

Many natural phenomena can be modeled by functions, such as the force of water against the face of a dam, explored in Exercise 85 on page 28.



Kunio Owaki/Corbis

Introduction to Functions

Many everyday phenomena involve pairs of quantities that are related to each other by some rule of correspondence. The mathematical term for such a rule of correspondence is a **relation.** Here are two examples.

1. The simple interest I earned on an investment of \$1000 for 1 year is related to the annual interest rate r by the formula $I = 1000r$.
2. The area A of a circle is related to its radius r by the formula $A = \pi r^2$.

Not all relations have simple mathematical formulas. For instance, people commonly match up NFL starting quarterbacks with touchdown passes, and hours of the day with temperature. In each of these cases, there is some relation that matches each item from one set with exactly one item from a different set. Such a relation is called a **function.**

Definition of a Function

A **function** f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

To help understand this definition, look at the function that relates the time of day to the temperature in Figure 1.14.

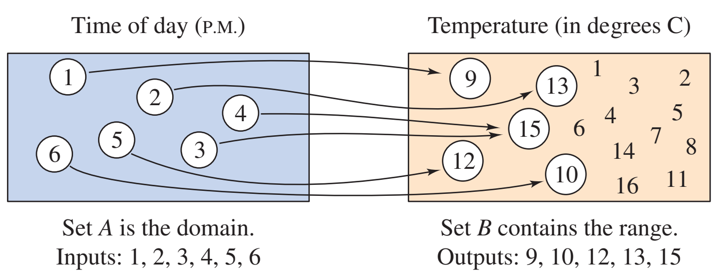


Figure 1.14

This function can be represented by the ordered pairs $\{(1, 9°), (2, 13°), (3, 15°), (4, 15°), (5, 12°), (6, 10°)\}$. In each ordered pair, the first coordinate (x-value) is the **input** and the second coordinate (y-value) is the **output.**

Characteristics of a Function from Set A to Set B

1. Each element of A must be matched with an element of B.
2. Some elements of B may not be matched with any element of A.
3. Two or more elements of A may be matched with the same element of B.
4. An element of A (the domain) cannot be matched with two different elements of B.

Library of Functions: Data Defined Function

Many functions do not have simple mathematical formulas, but are defined by real-life data. Such functions arise when you are using collections of data to model real-life applications. Functions can be represented in four ways.

1. *Verbally* by a sentence that describes how the input variables are related to the output variables

   Example: The input value x is the election year from 1952 to 2004 and the output value y is the elected president of the United States.

2. *Numerically* by a table or a list of ordered pairs that matches input values with output values

   Example: In the set of ordered pairs $\{(2, 34), (4, 40), (6, 45), (8, 50), (10, 54)\}$, the input value is the age of a male child in years and the output value is the height of the child in inches.

3. *Graphically* by points on a graph in a coordinate plane in which the input values are represented by the horizontal axis and the output values are represented by the vertical axis

   Example: See Figure 1.15.

4. *Algebraically* by an equation in two variables

   Example: The formula for temperature, $F = \frac{9}{5}C + 32$, where F is the temperature in degrees Fahrenheit and C is the temperature in degrees Celsius, is an equation that represents a function. You will see that it is often convenient to approximate data using a mathematical model or formula.

STUDY TIP

To determine whether or not a relation is a function, you must decide whether each input value is matched with exactly one output value. If any input value is matched with two or more output values, the relation is not a function.

Example 1 Testing for Functions

Decide whether the relation represents y as a function of x.

a.

Input, x	2	2	3	4	5
Output, y	11	10	8	5	1

b.

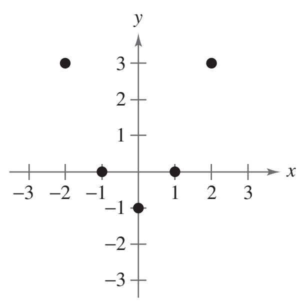


Figure 1.15

Prerequisite Skills

When plotting points in a coordinate plane, the x-coordinate is the directed distance from the y-axis to the point, and the y-coordinate is the directed distance from the x-axis to the point. To review point plotting, see Appendix B.1.

Solution

a. This table *does not* describe y as a function of x. The input value 2 is matched with two different y-values.

b. The graph in Figure 1.15 *does* describe y as a function of x. Each input value is matched with exactly one output value.

CHECKPOINT Now try Exercise 5.

STUDY TIP

Be sure you see that the *range* of a function is not the same as the use of *range* relating to the viewing window of a graphing utility.

In algebra, it is common to represent functions by equations or formulas involving two variables. For instance, the equation $y = x^2$ represents the variable y as a function of the variable x. In this equation, x is the **independent variable** and y is the **dependent variable.** The domain of the function is the set of all values taken on by the independent variable x, and the range of the function is the set of all values taken on by the dependent variable y.

Exploration

Use a graphing utility to graph $x^2 + y = 1$. Then use the graph to write a convincing argument that each x-value has at most one y-value.

Use a graphing utility to graph $-x + y^2 = 1$. (*Hint:* You will need to use two equations.) Does the graph represent y as a function of x? Explain.

Example 2 Testing for Functions Represented Algebraically

Which of the equations represent(s) y as a function of x?

a. $x^2 + y = 1$ **b.** $-x + y^2 = 1$

Solution

To determine whether y is a function of x, try to solve for y in terms of x.

a. Solving for y yields

$$x^2 + y = 1 \qquad \text{Write original equation.}$$

$$y = 1 - x^2. \qquad \text{Solve for } y.$$

Each value of x corresponds to exactly one value of y. So, y is a function of x.

b. Solving for y yields

$$-x + y^2 = 1 \qquad \text{Write original equation.}$$

$$y^2 = 1 + x \qquad \text{Add } x \text{ to each side.}$$

$$y = \pm\sqrt{1 + x}. \qquad \text{Solve for } y.$$

The $\pm$ indicates that for a given value of x there correspond two values of y. For instance, when $x = 3$, $y = 2$ or $y = -2$. So, y is not a function of x.

CHECKPOINT Now try Exercise 19.

Function Notation

When an equation is used to represent a function, it is convenient to name the function so that it can be referenced easily. For example, you know that the equation $y = 1 - x^2$ describes y as a function of x. Suppose you give this function the name "f." Then you can use the following **function notation.**

Input	*Output*	*Equation*
x	$f(x)$	$f(x) = 1 - x^2$

The symbol $f(x)$ is read as the *value of f at x* or simply *f of x*. The symbol $f(x)$ corresponds to the y-value for a given x. So, you can write $y = f(x)$. Keep in mind that f is the *name* of the function, whereas $f(x)$ is the *output value* of the function at the *input value* x. In function notation, the *input* is the independent variable and the *output* is the dependent variable. For instance, the function $f(x) = 3 - 2x$ has *function values* denoted by $f(-1)$, $f(0)$, and so on. To find these values, substitute the specified input values into the given equation.

For $x = -1$, $\quad f(-1) = 3 - 2(-1) = 3 + 2 = 5.$

For $x = 0$, $\quad f(0) = 3 - 2(0) = 3 - 0 = 3.$

TECHNOLOGY TIP

You can use a graphing utility to evaluate a function. Go to this textbook's *Online Study Center* and use the Evaluating an Algebraic Expression program. The program will prompt you for a value of x, and then evaluate the expression in the equation editor for that value of x. Try using the program to evaluate several different functions of x.

Although f is often used as a convenient function name and x is often used as the independent variable, you can use other letters. For instance,

$$f(x) = x^2 - 4x + 7, \quad f(t) = t^2 - 4t + 7, \quad \text{and} \quad g(s) = s^2 - 4s + 7$$

all define the same function. In fact, the role of the independent variable is that of a "placeholder." Consequently, the function could be written as

$$f(\quad) = (\quad)^2 - 4(\quad) + 7.$$

Example 3 Evaluating a Function

Let $g(x) = -x^2 + 4x + 1$. Find (a) $g(2)$, (b) $g(t)$, and (c) $g(x + 2)$.

Solution

a. Replacing x with 2 in $g(x) = -x^2 + 4x + 1$ yields the following.

$$g(2) = -(2)^2 + 4(2) + 1 = -4 + 8 + 1 = 5$$

b. Replacing x with t yields the following.

$$g(t) = -(t)^2 + 4(t) + 1 = -t^2 + 4t + 1$$

c. Replacing x with $x + 2$ yields the following.

$$\begin{aligned} g(x+2) &= -(x+2)^2 + 4(x+2) + 1 && \text{Substitute } x + 2 \text{ for } x. \\ &= -(x^2 + 4x + 4) + 4x + 8 + 1 && \text{Multiply.} \\ &= -x^2 - 4x - 4 + 4x + 8 + 1 && \text{Distributive Property} \\ &= -x^2 + 5 && \text{Simplify.} \end{aligned}$$

CHECKPOINT Now try Exercise 29.

In Example 3, note that $g(x + 2)$ is not equal to $g(x) + g(2)$. In general, $g(u + v) \neq g(u) + g(v)$.

Library of Parent Functions: Piecewise-Defined Function

A *piecewise-defined function* is a function that is defined by two or more equations over a specified domain. The *absolute value function* given by $f(x) = |x|$ can be written as a piecewise-defined function. The basic characteristics of the absolute value function are summarized below. A review of piecewise-defined functions can be found in the *Study Capsules.*

Graph of $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

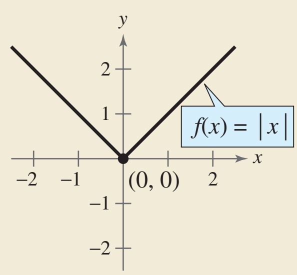
Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Intercept: $(0, 0)$

Decreasing on $(-\infty, 0)$

Increasing on $(0, \infty)$



Example 4 A Piecewise-Defined Function

Evaluate the function when $x = -1$ and $x = 0$.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

Solution

Because $x = -1$ is less than 0, use $f(x) = x^2 + 1$ to obtain

$$f(-1) = (-1)^2 + 1 = 2.$$

For $x = 0$, use $f(x) = x - 1$ to obtain

$$f(0) = 0 - 1 = -1.$$

CHECKPOINT Now try Exercise 37.

TECHNOLOGY TIP

Most graphing utilities can graph piecewise-defined functions. For instructions on how to enter a piecewise-defined function into your graphing utility, consult your user's manual. You may find it helpful to set your graphing utility to *dot mode* before graphing such functions.

The Domain of a Function

The domain of a function can be described explicitly or it can be *implied* by the expression used to define the function. The **implied domain** is the set of all real numbers for which the expression is defined. For instance, the function

$$f(x) = \frac{1}{x^2 - 4}$$ Domain excludes x-values that result in division by zero.

has an implied domain that consists of all real x other than $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined. Another common type of implied domain is that used to avoid even roots of negative numbers. For example, the function

$$f(x) = \sqrt{x}$$ Domain excludes x-values that result in even roots of negative numbers.

is defined only for $x \geq 0$. So, its implied domain is the interval $[0, \infty)$. In general, the domain of a function *excludes* values that would cause division by zero *or* result in the even root of a negative number.

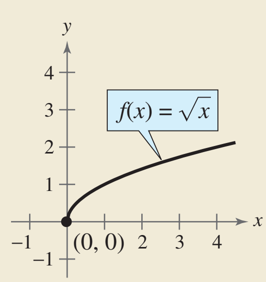
Exploration

Use a graphing utility to graph $y = \sqrt{4 - x^2}$. What is the domain of this function? Then graph $y = \sqrt{x^2 - 4}$. What is the domain of this function? Do the domains of these two functions overlap? If so, for what values?

Library of Parent Functions: Radical Function

Radical functions arise from the use of rational exponents. The most common radical function is the *square root function* given by $f(x) = \sqrt{x}$. The basic characteristics of the square root function are summarized below. A review of radical functions can be found in the *Study Capsules.*

Graph of $f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Increasing on $(0, \infty)$



STUDY TIP

Because the square root function is not defined for $x < 0$, you must be careful when analyzing the domains of complicated functions involving the square root symbol.

Example 5 Finding the Domain of a Function

Find the domain of each function.

a. f: $\{(-3, 0), (-1, 4), (0, 2), (2, 2), (4, -1)\}$

b. $g(x) = -3x^2 + 4x + 5$

c. $h(x) = \dfrac{1}{x + 5}$

d. Volume of a sphere: $V = \frac{4}{3}\pi r^3$

e. $k(x) = \sqrt{4 - 3x}$

Solution

a. The domain of f consists of all first coordinates in the set of ordered pairs.

$$\text{Domain} = \{-3, -1, 0, 2, 4\}$$

b. The domain of g is the set of all *real* numbers.

c. Excluding x-values that yield zero in the denominator, the domain of h is the set of all real numbers x except $x = -5$.

d. Because this function represents the volume of a sphere, the values of the radius r must be positive. So, the domain is the set of all real numbers r such that $r > 0$.

e. This function is defined only for x-values for which $4 - 3x \geq 0$. By solving this inequality, you will find that the domain of k is all real numbers that are less than or equal to $\frac{4}{3}$.

CHECKPOINT Now try Exercise 59.

Prerequisite Skills

In Example 5(e), $4 - 3x \geq 0$ is a *linear inequality*. To review solving of linear inequalities, see Appendix E.

In Example 5(d), note that the *domain of a function may be implied by the physical context*. For instance, from the equation $V = \frac{4}{3}\pi r^3$, you would have no reason to restrict r to positive values, but the physical context implies that a sphere cannot have a negative or zero radius.

For some functions, it may be easier to find the domain and range of the function by examining its graph.

Example 6 Finding the Domain and Range of a Function

Use a graphing utility to find the domain and range of the function

$$f(x) = \sqrt{9 - x^2}.$$

Solution

Graph the function as $y = \sqrt{9 - x^2}$, as shown in Figure 1.16. Using the *trace* feature of a graphing utility, you can determine that the x-values extend from -3 to 3 and the y-values extend from 0 to 3. So, the domain of the function f is all real numbers such that $-3 \leq x \leq 3$ and the range of f is all real numbers such that $0 \leq y \leq 3$.

CHECKPOINT Now try Exercise 63.

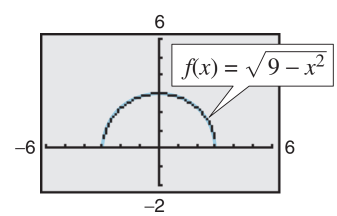


Figure 1.16

Applications

Example 7 Cellular Communications Employees

The number N (in thousands) of employees in the cellular communications industry in the United States increased in a linear pattern from 1998 to 2001 (see Figure 1.17). In 2002, the number dropped, then continued to increase through 2004 in a *different* linear pattern. These two patterns can be approximated by the function

$$N(t) = \begin{cases} 23.5t - 53.6, & 8 \le t \le 11 \\ 16.8t - 10.4, & 12 \le t \le 14 \end{cases}$$

where t represents the year, with $t = 8$ corresponding to 1998. Use this function to approximate the number of employees for each year from 1998 to 2004. (Source: Cellular Telecommunications & Internet Association)

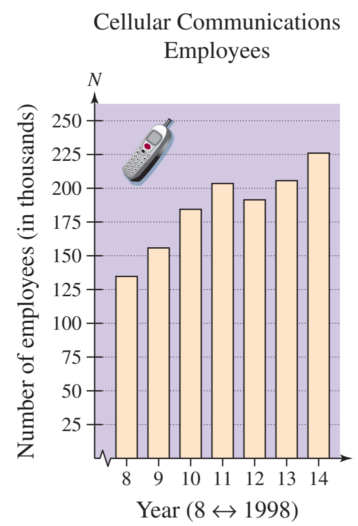


Figure 1.17

Solution

From 1998 to 2001, use $N(t) = 23.5t - 53.6$.

$$\underbrace{134.4}_{1998}, \quad \underbrace{157.9}_{1999}, \quad \underbrace{181.4}_{2000}, \quad \underbrace{204.9}_{2001}$$

From 2002 to 2004, use $N(t) = 16.8t - 10.4$.

$$\underbrace{191.2}_{2002}, \quad \underbrace{208.0}_{2003}, \quad \underbrace{224.8}_{2004}$$

CHECKPOINT Now try Exercise 83.

Example 8 The Path of a Baseball

A baseball is hit at a point 3 feet above the ground at a velocity of 100 feet per second and an angle of 45°. The path of the baseball is given by the function

$$f(x) = -0.0032x^2 + x + 3$$

where x and $f(x)$ are measured in feet. Will the baseball clear a 10-foot fence located 300 feet from home plate?

Algebraic Solution

The height of the baseball is a function of the horizontal distance from home plate. When $x = 300$, you can find the height of the baseball as follows.

$$f(x) = -0.0032x^2 + x + 3 \qquad \text{Write original function.}$$

$$f(300) = -0.0032(300)^2 + 300 + 3 \qquad \text{Substitute 300 for } x.$$

$$= 15 \qquad \text{Simplify.}$$

When $x = 300$, the height of the baseball is 15 feet, so the baseball will clear a 10-foot fence.

CHECKPOINT Now try Exercise 85.

Graphical Solution

Use a graphing utility to graph the function $y = -0.0032x^2 + x + 3$. Use the *value* feature or the *zoom* and *trace* features of the graphing utility to estimate that $y = 15$ when $x = 300$, as shown in Figure 1.18. So, the ball will clear a 10-foot fence.

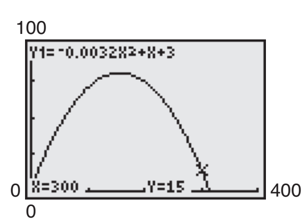


Figure 1.18

Difference Quotients

One of the basic definitions in calculus employs the ratio

$$\frac{f(x+h)-f(x)}{h}, \qquad h \neq 0.$$

This ratio is called a **difference quotient,** as illustrated in Example 9.

Example 9 Evaluating a Difference Quotient

algebra of calculus

For $f(x) = x^2 - 4x + 7$, find $\dfrac{f(x+h)-f(x)}{h}$.

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{[(x+h)^2 - 4(x+h) + 7] - (x^2 - 4x + 7)}{h}\\ &= \frac{x^2 + 2xh + h^2 - 4x - 4h + 7 - x^2 + 4x - 7}{h}\\ &= \frac{2xh + h^2 - 4h}{h}\\ &= \frac{h(2x+h-4)}{h} = 2x + h - 4, \;\; h \neq 0\end{aligned}$$

CHECKPOINT Now try Exercise 89.

STUDY TIP

Notice in Example 9 that h cannot be zero in the original expression. Therefore, you must restrict the domain of the simplified expression by adding $h \neq 0$ so that the simplified expression is equivalent to the original expression.

Summary of Function Terminology

Function: A **function** is a relationship between two variables such that to each value of the independent variable there corresponds exactly one value of the dependent variable.

Function Notation: $y = f(x)$

f is the *name* of the function.
y is the **dependent variable,** or output value.
x is the **independent variable,** or input value.
$f(x)$ is the *value of the function at* x.

Domain: The **domain** of a function is the set of all values (inputs) of the independent variable for which the function is defined. If x is in the domain of f, f is said to be *defined* at x. If x is not in the domain of f, f is said to be *undefined* at x.

Range: The **range** of a function is the set of all values (outputs) assumed by the dependent variable (that is, the set of all function values).

Implied Domain: If f is defined by an algebraic expression and the domain is not specified, the **implied domain** consists of all real numbers for which the expression is defined.

The symbol $\int$ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

1.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. A relation that assigns to each element x from a set of inputs, or ________ , exactly one element y in a set of outputs, or ________ , is called a ________ .

2. For an equation that represents y as a function of x, the ________ variable is the set of all x in the domain, and the ________ variable is the set of all y in the range.

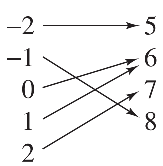
3. The function $f(x) = \begin{cases} x^2 - 4, & x \le 0 \\ 2x + 1, & x > 0 \end{cases}$ is an example of a ________ function.

4. If the domain of the function f is not given, then the set of values of the independent variable for which the expression is defined is called the ________ .

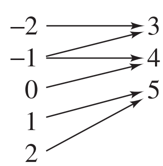
5. In calculus, one of the basic definitions is that of a ________ , given by $\dfrac{f(x+h) - f(x)}{h}$, $h \ne 0$.

In Exercises 1–4, does the relation describe a function? Explain your reasoning.

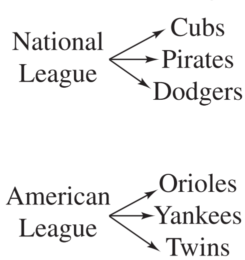
1. *Domain* *Range*



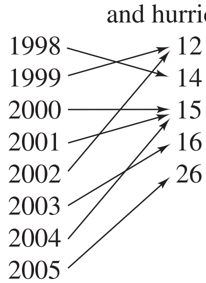
2. *Domain* *Range*



3. *Domain* *Range*



4. *Domain* (Year) *Range* (Number of North Atlantic tropical storms and hurricanes)



In Exercises 5–8, decide whether the relation represents y as a function of x. Explain your reasoning.

5.

Input, x	-3	-1	0	1	3
Output, y	-9	-1	0	1	9

6.

Input, x	0	1	2	1	0
Output, y	-4	-2	0	2	4

7.

Input, x	10	7	4	7	10
Output, y	3	6	9	12	15

8.

Input, x	0	3	9	12	15
Output, y	3	3	3	3	3

In Exercises 9 and 10, which sets of ordered pairs represent functions from A to B? Explain.

9. $A = \{0, 1, 2, 3\}$ and $B = \{-2, -1, 0, 1, 2\}$

(a) $\{(0, 1), (1, -2), (2, 0), (3, 2)\}$

(b) $\{(0, -1), (2, 2), (1, -2), (3, 0), (1, 1)\}$

(c) $\{(0, 0), (1, 0), (2, 0), (3, 0)\}$

(d) $\{(0, 2), (3, 0), (1, 1)\}$

10. $A = \{a, b, c\}$ and $B = \{0, 1, 2, 3\}$

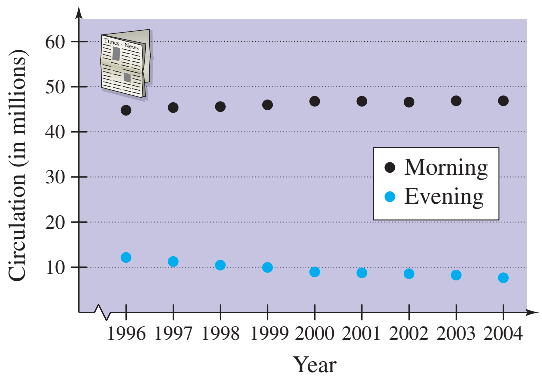
(a) $\{(a, 1), (c, 2), (c, 3), (b, 3)\}$

(b) $\{(a, 1), (b, 2), (c, 3)\}$

(c) $\{(1, a), (0, a), (2, c), (3, b)\}$

(d) $\{(c, 0), (b, 0), (a, 3)\}$

Circulation of Newspapers **In Exercises 11 and 12, use the graph, which shows the circulation (in millions) of daily newspapers in the United States.** (Source: Editor & Publisher Company)



11. Is the circulation of morning newspapers a function of the year? Is the circulation of evening newspapers a function of the year? Explain.

12. Let $f(x)$ represent the circulation of evening newspapers in year x. Find $f(2004)$.

In Exercises 13–24, determine whether the equation represents y as a function of x.

13. $x^2 + y^2 = 4$ **14.** $x = y^2 + 1$

15. $y = \sqrt{x^2 - 1}$ **16.** $y = \sqrt{x + 5}$

17. $2x + 3y = 4$ **18.** $x = -y + 5$

19. $y^2 = x^2 - 1$ **20.** $x + y^2 = 3$

21. $y = |4 - x|$ **22.** $|y| = 4 - x$

23. $x = -7$ **24.** $y = 8$

In Exercises 25 and 26, fill in the blanks using the specified function and the given values of the independent variable. Simplify the result.

25. $f(x) = \dfrac{1}{x+1}$

(a) $f(4) = \dfrac{1}{(\quad) + 1}$ (b) $f(0) = \dfrac{1}{(\quad) + 1}$

(c) $f(4t) = \dfrac{1}{(\quad) + 1}$ (d) $f(x + c) = \dfrac{1}{(\quad) + 1}$

26. $g(x) = x^2 - 2x$

(a) $g(2) = (\quad)^2 - 2(\quad)$

(b) $g(-3) = (\quad)^2 - 2(\quad)$

(c) $g(t + 1) = (\quad)^2 - 2(\quad)$

(d) $g(x + c) = (\quad)^2 - 2(\quad)$

In Exercises 27–42, evaluate the function at each specified value of the independent variable and simplify.

27. $f(t) = 3t + 1$

(a) $f(2)$ (b) $f(-4)$ (c) $f(t + 2)$

28. $g(y) = 7 - 3y$

(a) $g(0)$ (b) $g\left(\frac{7}{3}\right)$ (c) $g(s + 2)$

29. $h(t) = t^2 - 2t$

(a) $h(2)$ (b) $h(1.5)$ (c) $h(x + 2)$

30. $V(r) = \frac{4}{3}\pi r^3$

(a) $V(3)$ (b) $V\left(\frac{3}{2}\right)$ (c) $V(2r)$

31. $f(y) = 3 - \sqrt{y}$

(a) $f(4)$ (b) $f(0.25)$ (c) $f(4x^2)$

32. $f(x) = \sqrt{x + 8} + 2$

(a) $f(-8)$ (b) $f(1)$ (c) $f(x - 8)$

33. $q(x) = \dfrac{1}{x^2 - 9}$

(a) $q(0)$ (b) $q(3)$ (c) $q(y + 3)$

34. $q(t) = \dfrac{2t^2 + 3}{t^2}$

(a) $q(2)$ (b) $q(0)$ (c) $q(-x)$

35. $f(x) = \dfrac{|x|}{x}$

(a) $f(3)$ (b) $f(-3)$ (c) $f(t)$

36. $f(x) = |x| + 4$

(a) $f(4)$ (b) $f(-4)$ (c) $f(t)$

37. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$

(a) $f(-1)$ (b) $f(0)$ (c) $f(2)$

38. $f(x) = \begin{cases} 2x + 5, & x \leq 0 \\ 2 - x^2, & x > 0 \end{cases}$

(a) $f(-2)$ (b) $f(0)$ (c) $f(1)$

39. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$

(a) $f(-2)$ (b) $f(1)$ (c) $f(2)$

40. $f(x) = \begin{cases} x^2 - 4, & x \leq 0 \\ 1 - 2x^2, & x > 0 \end{cases}$

(a) $f(-2)$ (b) $f(0)$ (c) $f(1)$

41. $f(x) = \begin{cases} x + 2, & x < 0 \\ 4, & 0 \leq x < 2 \\ x^2 + 1, & x \geq 2 \end{cases}$

(a) $f(-2)$ (b) $f(1)$ (c) $f(4)$

42. $f(x) = \begin{cases} 5 - 2x, & x < 0 \\ 5, & 0 \le x < 1 \\ 4x + 1, & x \ge 1 \end{cases}$

(a) $f(-2)$ (b) $f(\frac{1}{2})$ (c) $f(1)$

In Exercises 43–46, complete the table.

43. $h(t) = \frac{1}{2}|t + 3|$

t	-5	-4	-3	-2	-1
$h(t)$					

44. $f(s) = \dfrac{|s - 2|}{s - 2}$

s	0	1	$\frac{3}{2}$	$\frac{5}{2}$	4
$f(s)$					

45. $f(x) = \begin{cases} -\frac{1}{2}x + 4, & x \le 0 \\ (x - 2)^2, & x > 0 \end{cases}$

x	-2	-1	0	1	2
$f(x)$					

46. $h(x) = \begin{cases} 9 - x^2, & x < 3 \\ x - 3, & x \ge 3 \end{cases}$

x	1	2	3	4	5
$h(x)$					

In Exercises 47–50, find all real values of x such that $f(x) = 0$.

47. $f(x) = 15 - 3x$

48. $f(x) = 5x + 1$

49. $f(x) = \dfrac{3x - 4}{5}$

50. $f(x) = \dfrac{2x - 3}{7}$

In Exercises 51 and 52, find the value(s) of x for which $f(x) = g(x)$.

51. $f(x) = x^2, \quad g(x) = x + 2$

52. $f(x) = x^2 + 2x + 1, \quad g(x) = 7x - 5$

In Exercises 53–62, find the domain of the function.

53. $f(x) = 5x^2 + 2x - 1$

54. $g(x) = 1 - 2x^2$

55. $h(t) = \dfrac{4}{t}$

56. $s(y) = \dfrac{3y}{y + 5}$

57. $f(x) = \sqrt[3]{x - 4}$

58. $f(x) = \sqrt[4]{x^2 + 3x}$

59. $g(x) = \dfrac{1}{x} - \dfrac{3}{x + 2}$

60. $h(x) = \dfrac{10}{x^2 - 2x}$

61. $g(y) = \dfrac{y + 2}{\sqrt{y - 10}}$

62. $f(x) = \dfrac{\sqrt{x + 6}}{6 + x}$

In Exercises 63–66, use a graphing utility to graph the function. Find the domain and range of the function.

63. $f(x) = \sqrt{4 - x^2}$

64. $f(x) = \sqrt{x^2 + 1}$

65. $g(x) = |2x + 3|$

66. $g(x) = |x - 5|$

In Exercises 67–70, assume that the domain of f is the set $A = \{-2, -1, 0, 1, 2\}$. Determine the set of ordered pairs representing the function f.

67. $f(x) = x^2$

68. $f(x) = x^2 - 3$

69. $f(x) = |x| + 2$

70. $f(x) = |x + 1|$

71. ***Geometry*** Write the area A of a circle as a function of its circumference C.

72. ***Geometry*** Write the area A of an equilateral triangle as a function of the length s of its sides.

73. ***Exploration*** The cost per unit to produce a radio model is \$60. The manufacturer charges \$90 per unit for orders of 100 or less. To encourage large orders, the manufacturer reduces the charge by \$0.15 per radio for each unit ordered in excess of 100 (for example, there would be a charge of \$87 per radio for an order size of 120).

(a) The table shows the profit P (in dollars) for various numbers of units ordered, x. Use the table to estimate the maximum profit.

Units, x	Profit, P
110	3135
120	3240
130	3315
140	3360
150	3375
160	3360
170	3315

(b) Plot the points (x, P) from the table in part (a). Does the relation defined by the ordered pairs represent P as a function of x?

(c) If P is a function of x, write the function and determine its domain.

74. ***Exploration*** An open box of maximum volume is to be made from a square piece of material, 24 centimeters on a side, by cutting equal squares from the corners and turning up the sides (see figure).

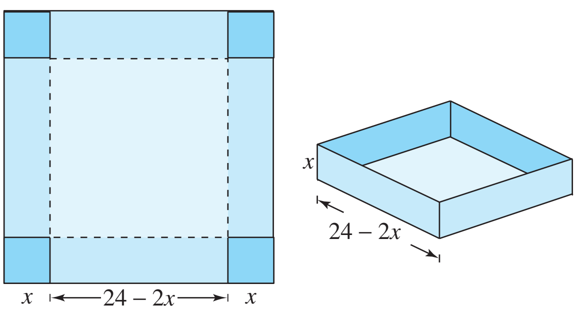
(a) The table shows the volume V (in cubic centimeters) of the box for various heights x (in centimeters). Use the table to estimate the maximum volume.

Height, x	Volume, V
1	484
2	800
3	972
4	1024
5	980
6	864

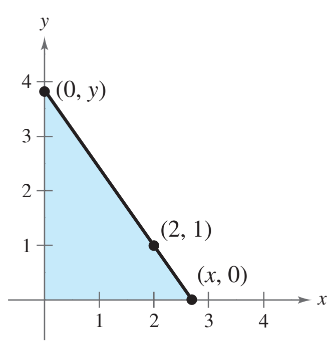
(b) Plot the points (x, V) from the table in part (a). Does the relation defined by the ordered pairs represent V as a function of x?

(c) If V is a function of x, write the function and determine its domain.

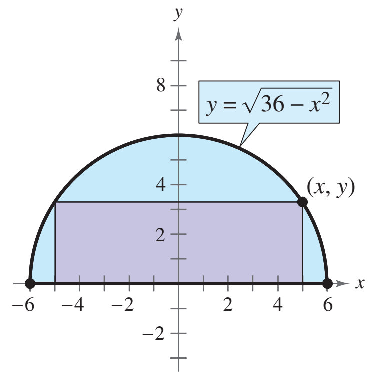
(d) Use a graphing utility to plot the point from the table in part (a) with the function from part (c). How closely does the function represent the data? Explain.



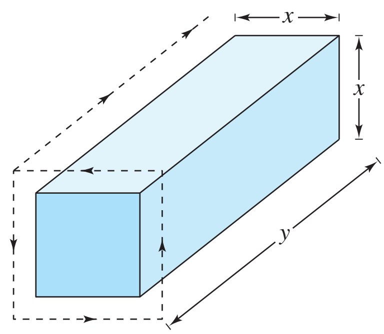
75. ***Geometry*** A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(2, 1)$ (see figure). Write the area A of the triangle as a function of x, and determine the domain of the function.



76. ***Geometry*** A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{36 - x^2}$ (see figure). Write the area A of the rectangle as a function of x, and determine the domain of the function.



77. ***Postal Regulations*** A rectangular package to be sent by the U.S. Postal Service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure).



(a) Write the volume V of the package as a function of x. What is the domain of the function?

(b) Use a graphing utility to graph the function. Be sure to use an appropriate viewing window.

(c) What dimensions will maximize the volume of the package? Explain.

78. ***Cost, Revenue, and Profit*** A company produces a toy for which the variable cost is \$12.30 per unit and the fixed costs are \$98,000. The toy sells for \$17.98. Let x be the number of units produced and sold.

(a) The total cost for a business is the sum of the variable cost and the fixed costs. Write the total cost C as a function of the number of units produced.

(b) Write the revenue R as a function of the number of units sold.

(c) Write the profit P as a function of the number of units sold. (*Note:* $P = R - C$.)

Revenue **In Exercises 79–82, use the table, which shows the monthly revenue y (in thousands of dollars) of a landscaping business for each month of 2006, with $x = 1$ representing January.**

Month, x	Revenue, y
1	5.2
2	5.6
3	6.6
4	8.3
5	11.5
6	15.8
7	12.8
8	10.1
9	8.6
10	6.9
11	4.5
12	2.7

A mathematical model that represents the data is

$$f(x) = \begin{cases} -1.97x + 26.3 \\ 0.505x^2 - 1.47x + 6.3 \end{cases}.$$

79. What is the domain of each part of the piecewise-defined function? Explain your reasoning.

80. Use the mathematical model to find $f(5)$. Interpret your result in the context of the problem.

81. Use the mathematical model to find $f(11)$. Interpret your result in the context of the problem.

82. How do the values obtained from the model in Exercises 80 and 81 compare with the actual data values?

83. ***Motor Vehicles*** The numbers n (in billions) of miles traveled by vans, pickup trucks, and sport utility vehicles in the United States from 1990 to 2003 can be approximated by the model

$$n(t) = \begin{cases} -6.13t^2 + 75.8t + 577, & 0 \le t \le 6 \\ 24.9t + 672, & 6 < t \le 13 \end{cases}$$

where t represents the year, with $t = 0$ corresponding to 1990. Use the *table* feature of a graphing utility to approximate the number of miles traveled by vans, pickup trucks, and sport utility vehicles for each year from 1990 to 2003. (Source: U.S. Federal Highway Administration)

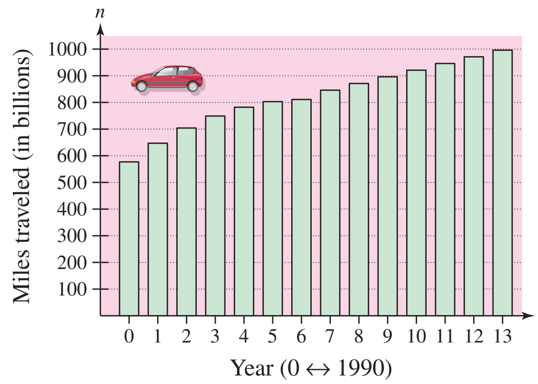


Figure for 83

84. ***Transportation*** For groups of 80 or more people, a charter bus company determines the rate per person according to the formula

Rate $= 8 - 0.05(n - 80),\ n \ge 80$

where the rate is given in dollars and n is the number of people.

(a) Write the revenue R of the bus company as a function of n.

(b) Use the function from part (a) to complete the table. What can you conclude?

n	90	100	110	120	130	140	150
$R(n)$							

(c) Use a graphing utility to graph R and determine the number of people that will produce a maximum revenue. Compare the result with your conclusion from part (b).

85. ***Physics*** The force F (in tons) of water against the face of a dam is estimated by the function

$F(y) = 149.76\sqrt{10}y^{5/2}$

where y is the depth of the water (in feet).

(a) Complete the table. What can you conclude from it?

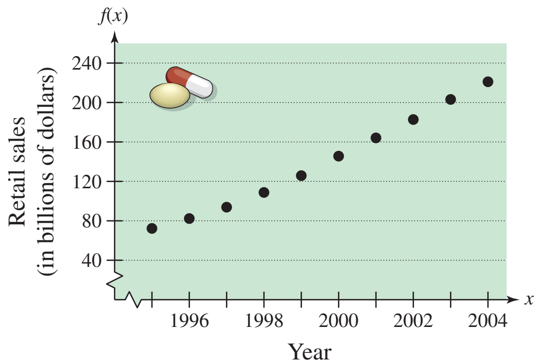
y	5	10	20	30	40
$F(y)$					

(b) Use a graphing utility to graph the function. Describe your viewing window.

(c) Use the table to approximate the depth at which the force against the dam is 1,000,000 tons. How could you find a better estimate?

(d) Verify your answer in part (c) graphically.

86. ***Data Analysis*** The graph shows the retail sales (in billions of dollars) of prescription drugs in the United States from 1995 through 2004. Let $f(x)$ represent the retail sales in year x. (Source: National Association of Chain Drug Stores)



(a) Find $f(2000)$.

(b) Find $\dfrac{f(2004) - f(1995)}{2004 - 1995}$

and interpret the result in the context of the problem.

(c) An approximate model for the function is

$P(t) = -0.0982t^3 + 3.365t^2 - 18.85t + 94.8,$
$5 \le t \le 14$

where P is the retail sales (in billions of dollars) and t represents the year, with $t = 5$ corresponding to 1995. Complete the table and compare the results with the data in the graph.

t	5	6	7	8	9	10	11	12	13	14
$P(t)$										

(d) Use a graphing utility to graph the model and the data in the same viewing window. Comment on the validity of the model.

In Exercises 87–92, find the difference quotient and simplify your answer.

87. $f(x) = 2x, \quad \dfrac{f(x + c) - f(x)}{c}, \quad c \neq 0$

88. $g(x) = 3x - 1, \quad \dfrac{g(x + h) - g(x)}{h}, \quad h \neq 0$

89. $f(x) = x^2 - x + 1, \quad \dfrac{f(2 + h) - f(2)}{h}, \quad h \neq 0$

90. $f(x) = x^3 + x, \quad \dfrac{f(x + h) - f(x)}{h}, \quad h \neq 0$

91. $f(t) = \dfrac{1}{t}, \quad \dfrac{f(t) - f(1)}{t - 1}, \quad t \neq 1$

92. $f(x) = \dfrac{4}{x + 1}, \quad \dfrac{f(x) - f(7)}{x - 7}, \quad x \neq 7$

Synthesis

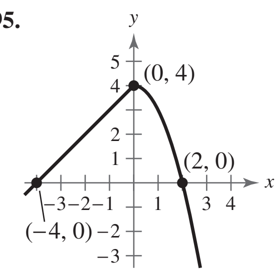
True or False? **In Exercises 93 and 94, determine whether the statement is true or false. Justify your answer.**

93. The domain of the function $f(x) = x^4 - 1$ is $(-\infty, \infty)$, and the range of $f(x)$ is $(0, \infty)$.

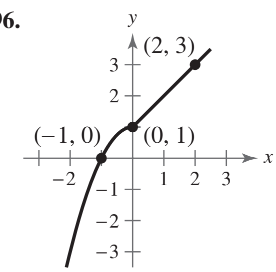
94. The set of ordered pairs $\{(-8, -2), (-6, 0), (-4, 0), (-2, 2), (0, 4), (2, -2)\}$ represents a function.

Library of Parent Functions **In Exercises 95–98, write a piecewise-defined function for the graph shown.**

95.



96.



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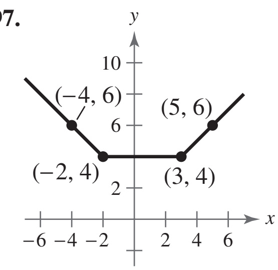
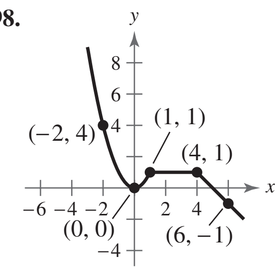


98.



99. ***Writing*** In your own words, explain the meanings of *domain* and *range*.

100. ***Think About It*** Describe an advantage of function notation.

Skills Review

In Exercises 101–104, perform the operation and simplify.

101. $12 - \dfrac{4}{x + 2}$

102. $\dfrac{3}{x^2 + x - 20} + \dfrac{x}{x^2 + 4x - 5}$

103. $\dfrac{2x^3 + 11x^2 - 6x}{5x} \cdot \dfrac{x + 10}{2x^2 + 5x - 3}$

104. $\dfrac{x + 7}{2(x - 9)} \div \dfrac{x - 7}{2(x - 9)}$

The symbol $\int$ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

1.3 Graphs of Functions

What you should learn

- Find the domains and ranges of functions and use the Vertical Line Test for functions.
- Determine intervals on which functions are increasing, decreasing, or constant.
- Determine relative maximum and relative minimum values of functions.
- Identify and graph step functions and other piecewise-defined functions.
- Identify even and odd functions.

Why you should learn it

Graphs of functions provide a visual relationship between two variables. For example, in Exercise 88 on page 40, you will use the graph of a step function to model the cost of sending a package.

Stephen Chernin/Getty Images

The Graph of a Function

In Section 1.2, functions were represented graphically by points on a graph in a coordinate plane in which the input values are represented by the horizontal axis and the output values are represented by the vertical axis. The **graph of a function** f is the collection of ordered pairs $(x, f(x))$ such that x is in the domain of f. As you study this section, remember the geometric interpretations of x and $f(x)$.

$x =$ the directed distance from the y-axis

$f(x) =$ the directed distance from the x-axis

Example 1 shows how to use the graph of a function to find the domain and range of the function.

Example 1 Finding the Domain and Range of a Function

Use the graph of the function f shown in Figure 1.19 to find (a) the domain of f, (b) the function values $f(-1)$ and $f(2)$, and (c) the range of f.

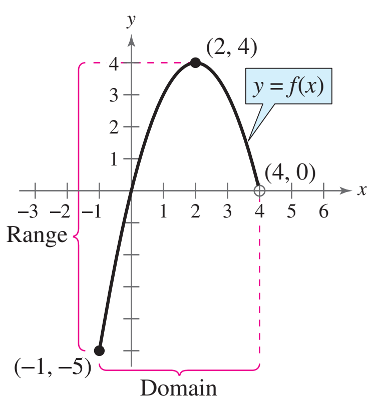


Figure 1.19

Solution

a. The closed dot at $(-1, -5)$ indicates that $x = -1$ is in the domain of f, whereas the open dot at $(4, 0)$ indicates that $x = 4$ is not in the domain. So, the domain of f is all x in the interval $[-1, 4)$.

b. Because $(-1, -5)$ is a point on the graph of f, it follows that

$$f(-1) = -5.$$

Similarly, because $(2, 4)$ is a point on the graph of f, it follows that

$$f(2) = 4.$$

c. Because the graph does not extend below $f(-1) = -5$ or above $f(2) = 4$, the range of f is the interval $[-5, 4]$.

CHECKPOINT Now try Exercise 3.

STUDY TIP

The use of dots (open or closed) at the extreme left and right points of a graph indicates that the graph does not extend beyond these points. If no such dots are shown, assume that the graph extends beyond these points.

Example 2 Finding the Domain and Range of a Function

Find the domain and range of

$$f(x) = \sqrt{x - 4}.$$

Algebraic Solution

Because the expression under a radical cannot be negative, the domain of $f(x) = \sqrt{x - 4}$ is the set of all real numbers such that $x - 4 \geq 0$. Solve this linear inequality for x as follows. (For help with solving linear inequalities, see Appendix E.)

$$x - 4 \geq 0 \qquad \text{Write original inequality.}$$

$$x \geq 4 \qquad \text{Add 4 to each side.}$$

So, the domain is the set of all real numbers greater than or equal to 4. Because the value of a radical expression is never negative, the range of $f(x) = \sqrt{x - 4}$ is the set of all nonnegative real numbers.

CHECKPOINT Now try Exercise 7.

Graphical Solution

Use a graphing utility to graph the equation $y = \sqrt{x - 4}$, as shown in Figure 1.20. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from 4 to the right. When x is greater than or equal to 4, the expression under the radical is nonnegative. So, you can conclude that the domain is the set of all real numbers greater than or equal to 4. From the graph, you can see that the y-coordinates of points on the graph extend from 0 upwards. So you can estimate the range to be the set of all nonnegative real numbers.

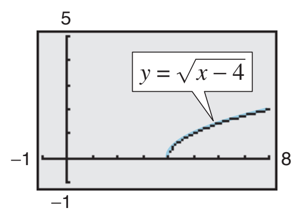


Figure 1.20

By the definition of a function, at most one y-value corresponds to a given x-value. It follows, then, that a vertical line can intersect the graph of a function at most once. This leads to the **Vertical Line Test** for functions.

Vertical Line Test for Functions

A set of points in a coordinate plane is the graph of y as a function of x if and only if no vertical line intersects the graph at more than one point.

Example 3 Vertical Line Test for Functions

Use the Vertical Line Test to decide whether the graphs in Figure 1.21 represent y as a function of x.

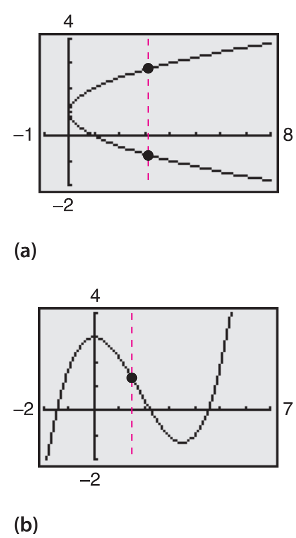


Figure 1.21

Solution

a. This is *not* a graph of y as a function of x because you can find a vertical line that intersects the graph twice.

b. This *is* a graph of y as a function of x because every vertical line intersects the graph at most once.

CHECKPOINT Now try Exercise 17.

Increasing and Decreasing Functions

The more you know about the graph of a function, the more you know about the function itself. Consider the graph shown in Figure 1.22. Moving from *left to right*, this graph falls from $x = -2$ to $x = 0$, is constant from $x = 0$ to $x = 2$, and rises from $x = 2$ to $x = 4$.

Increasing, Decreasing, and Constant Functions

A function f is **increasing** on an interval if, for any x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) < f(x_2).$$

A function f is **decreasing** on an interval if, for any x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) > f(x_2).$$

A function f is **constant** on an interval if, for any x_1 and x_2 in the interval,

$$f(x_1) = f(x_2).$$

TECHNOLOGY TIP

Most graphing utilities are designed to graph functions of x more easily than other types of equations. For instance, the graph shown in Figure 1.23(a) represents the equation $x - (y - 1)^2 = 0$. To use a graphing utility to duplicate this graph you must first solve the equation for y to obtain $y = 1 \pm \sqrt{x}$, and then graph the two equations $y_1 = 1 + \sqrt{x}$ and $y_2 = 1 - \sqrt{x}$ in the same viewing window.

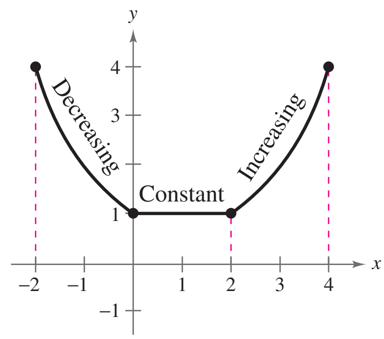


Figure 1.22

Example 4 Increasing and Decreasing Functions

In Figure 1.23, determine the open intervals on which each function is increasing, decreasing, or constant.

Solution

a. Although it might appear that there is an interval in which this function is constant, you can see that if $x_1 < x_2$, then $(x_1)^3 < (x_2)^3$, which implies that $f(x_1) < f(x_2)$. So, the function is increasing over the entire real line.

b. This function is increasing on the interval $(-\infty, -1)$, decreasing on the interval $(-1, 1)$, and increasing on the interval $(1, \infty)$.

c. This function is increasing on the interval $(-\infty, 0)$, constant on the interval $(0, 2)$, and decreasing on the interval $(2, \infty)$.

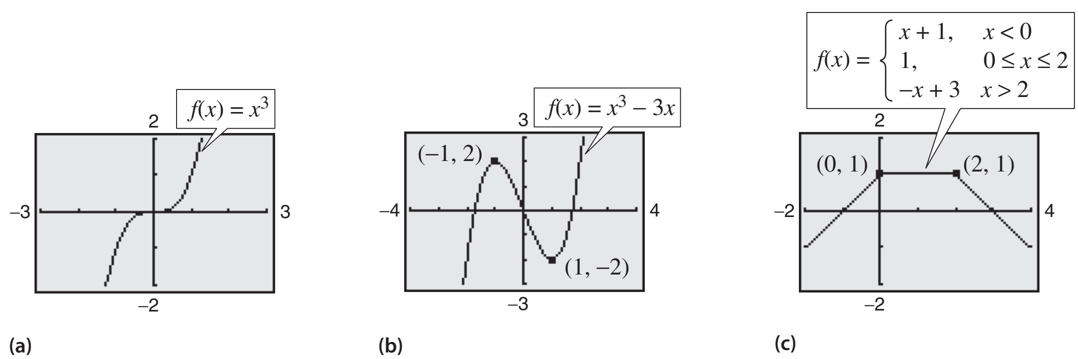


Figure 1.23

CHECKPOINT Now try Exercise 21.

Relative Minimum and Maximum Values

The points at which a function changes its increasing, decreasing, or constant behavior are helpful in determining the relative maximum or relative minimum values of the function.

Definitions of Relative Minimum and Relative Maximum

A function value $f(a)$ is called a **relative minimum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \le f(x).$$

A function value $f(a)$ is called a **relative maximum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \ge f(x).$$

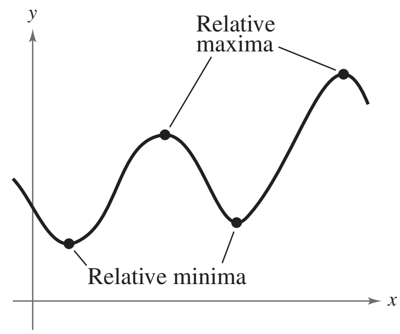


Figure 1.24

Figure 1.24 shows several different examples of relative minima and relative maxima. In Section 2.1, you will study a technique for finding the *exact points* at which a second-degree polynomial function has a relative minimum or relative maximum. For the time being, however, you can use a graphing utility to find reasonable approximations of these points.

Example 5 Approximating a Relative Minimum

Use a graphing utility to approximate the relative minimum of the function given by $f(x) = 3x^2 - 4x - 2$.

Solution

The graph of f is shown in Figure 1.25. By using the *zoom* and *trace* features of a graphing utility, you can estimate that the function has a relative minimum at the point

$(0.67, -3.33)$. See Figure 1.26.

Later, in Section 2.1, you will be able to determine that the exact point at which the relative minimum occurs is $\left(\frac{2}{3}, -\frac{10}{3}\right)$.

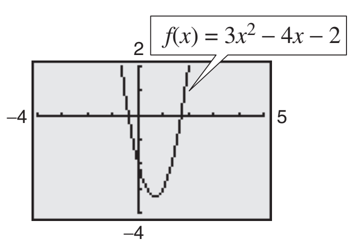


Figure 1.25

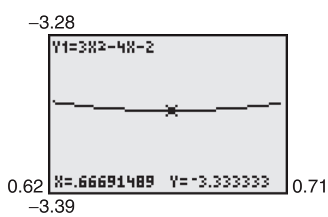


Figure 1.26

CHECKPOINT Now try Exercise 31.

TECHNOLOGY TIP Some graphing utilities have built-in programs that will find minimum or maximum values. These features are demonstrated in Example 6.

TECHNOLOGY TIP

When you use a graphing utility to estimate the x- and y-values of a relative minimum or relative maximum, the *zoom* feature will often produce graphs that are nearly flat, as shown in Figure 1.26. To overcome this problem, you can manually change the vertical setting of the viewing window. The graph will stretch vertically if the values of Ymin and Ymax are closer together.

Example 6 Approximating Relative Minima and Maxima

Use a graphing utility to approximate the relative minimum and relative maximum of the function given by $f(x) = -x^3 + x$.

Solution

The graph of f is shown in Figure 1.27. By using the *zoom* and *trace* features or the *minimum* and *maximum* features of the graphing utility, you can estimate that the function has a relative minimum at the point

$(-0.58, -0.38)$ See Figure 1.28.

and a relative maximum at the point

$(0.58, 0.38)$. See Figure 1.29.

If you take a course in calculus, you will learn a technique for finding the exact points at which this function has a relative minimum and a relative maximum.

CHECKPOINT Now try Exercise 33.

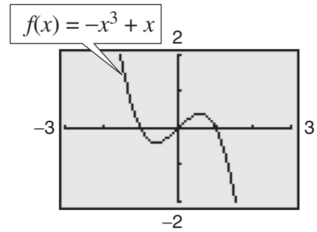


Figure 1.27

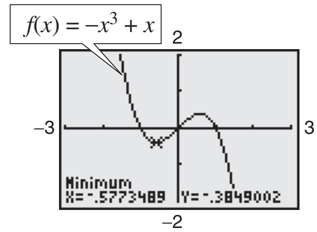


Figure 1.28

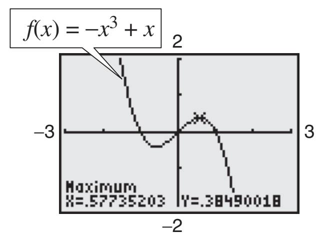


Figure 1.29

Example 7 Temperature

During a 24-hour period, the temperature y (in degrees Fahrenheit) of a certain city can be approximated by the model

$$y = 0.026x^3 - 1.03x^2 + 10.2x + 34, \qquad 0 \le x \le 24$$

where x represents the time of day, with $x = 0$ corresponding to 6 A.M. Approximate the maximum and minimum temperatures during this 24-hour period.

Solution

To solve this problem, graph the function as shown in Figure 1.30. Using the *zoom* and *trace* features or the *maximum* feature of a graphing utility, you can determine that the maximum temperature during the 24-hour period was approximately 64°F. This temperature occurred at about 12:36 P.M. ($x \approx 6.6$), as shown in Figure 1.31. Using the *zoom* and *trace* features or the *minimum* feature, you can determine that the minimum temperature during the 24-hour period was approximately 34°F, which occurred at about 1:48 A.M. ($x \approx 19.8$), as shown in Figure 1.32.

TECHNOLOGY SUPPORT

For instructions on how to use the *minimum* and *maximum* features, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

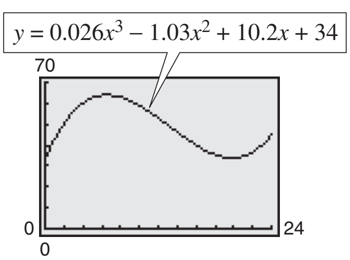


Figure 1.30

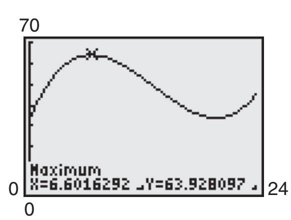


Figure 1.31

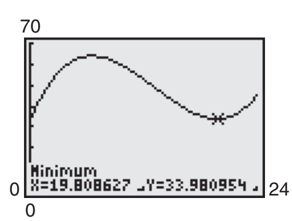


Figure 1.32

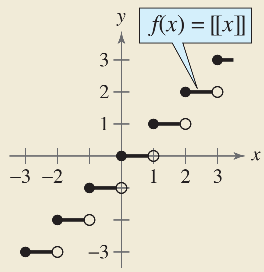
CHECKPOINT Now try Exercise 91.

Graphing Step Functions and Piecewise-Defined Functions

Library of Parent Functions: Greatest Integer Function

The *greatest integer function*, denoted by $[\![x]\!]$ and defined as the greatest integer less than or equal to x, has an infinite number of breaks or steps—one at each integer value in its domain. The basic characteristics of the greatest integer function are summarized below. A review of the greatest integer function can be found in the *Study Capsules.*

Graph of $f(x) = [\![x]\!]$
Domain: $(-\infty, \infty)$
Range: the set of integers
x-intercepts: in the interval $[0, 1)$
y-intercept: $(0, 0)$
Constant between each pair of consecutive integers
Jumps vertically one unit at each integer value



Could you describe the greatest integer function using a piecewise-defined function? How does the graph of the greatest integer function differ from the graph of a line with a slope of zero?

TECHNOLOGY TIP

Most graphing utilities display graphs in *connected mode*, which means that the graph has no breaks. When you are sketching graphs that do have breaks, it is better to use *dot mode*. Graph the greatest integer function [often called Int (x)] in *connected* and *dot modes*, and compare the two results.

Because of the vertical jumps described above, the greatest integer function is an example of a **step function** whose graph resembles a set of stairsteps. Some values of the greatest integer function are as follows.

$$[\![-1]\!] = (\text{greatest integer} \le -1) = -1$$

$$\left[\!\!\left[\tfrac{1}{10}\right]\!\!\right] = \left(\text{greatest integer} \le \tfrac{1}{10}\right) = 0$$

$$[\![1.5]\!] = (\text{greatest integer} \le 1.5) = 1$$

In Section 1.2, you learned that a piecewise-defined function is a function that is defined by two or more equations over a specified domain. To sketch the graph of a piecewise-defined function, you need to sketch the graph of each equation on the appropriate portion of the domain.

Example 8 Graphing a Piecewise-Defined Function

Sketch the graph of $f(x) = \begin{cases} 2x + 3, & x \le 1 \\ -x + 4, & x > 1 \end{cases}$ by hand.

Solution

This piecewise-defined function is composed of two linear functions. At and to the left of $x = 1$, the graph is the line given by $y = 2x + 3$. To the right of $x = 1$, the graph is the line given by $y = -x + 4$ (see Figure 1.33). Notice that the point $(1, 5)$ is a solid dot and the point $(1, 3)$ is an open dot. This is because $f(1) = 5$.

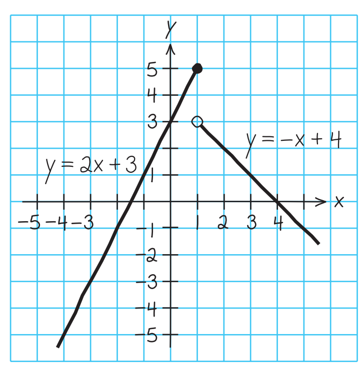
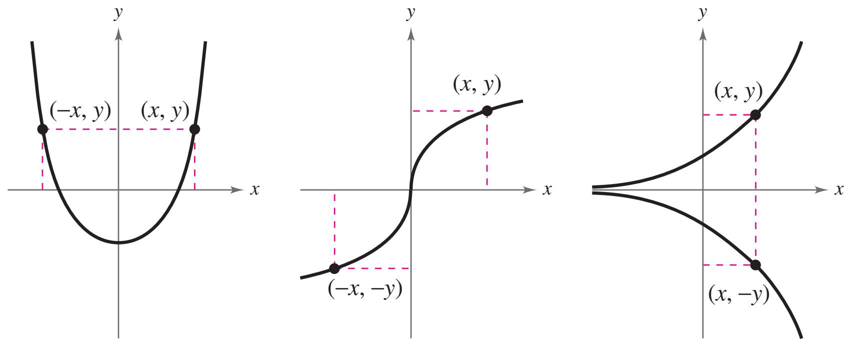


Figure 1.33

CHECKPOINT Now try Exercise 43.

Even and Odd Functions

A graph has *symmetry with respect to the y-axis* if whenever (x, y) is on the graph, so is the point $(-x, y)$. A graph has *symmetry with respect to the origin* if whenever (x, y) is on the graph, so is the point $(-x, -y)$. A graph has *symmetry with respect to the x-axis* if whenever (x, y) is on the graph, so is the point $(x, -y)$. A function whose graph is symmetric with respect to the y-axis is an **even function.** A function whose graph is symmetric with respect to the origin is an **odd function.** A graph that is symmetric with respect to the x-axis is not the graph of a function (except for the graph of $y = 0$). These three types of symmetry are illustrated in Figure 1.34.



Symmetric to y-axis
Even function

Symmetric to origin
Odd function

Symmetric to x-axis
Not a function

Figure 1.34

Test for Even and Odd Functions

A function f is **even** if, for each x in the domain of f, $f(-x) = f(x)$.
A function f is **odd** if, for each x in the domain of f, $f(-x) = -f(x)$.

Example 9 Testing for Evenness and Oddness

Is the function given by $f(x) = |x|$ even, odd, or neither?

Algebraic Solution

This function is even because

$$\begin{aligned} f(-x) &= |-x| \\ &= |x| \\ &= f(x). \end{aligned}$$

Graphical Solution

Use a graphing utility to enter $y = |x|$ in the *equation editor*, as shown in Figure 1.35. Then graph the function using a standard viewing window, as shown in Figure 1.36. You can see that the graph appears to be symmetric about the y-axis. So, the function is even.

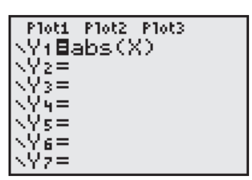


Figure 1.35

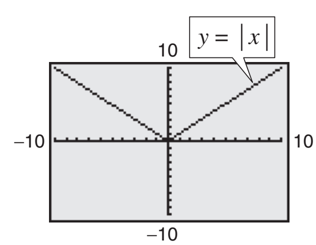


Figure 1.36

CHECKPOINT Now try Exercise 59.

Example 10 Even and Odd Functions

Determine whether each function is even, odd, or neither.

a. $g(x) = x^3 - x$

b. $h(x) = x^2 + 1$

c. $f(x) = x^3 - 1$

Algebraic Solution

a. This function is odd because

$$\begin{aligned} g(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -(x^3 - x) \\ &= -g(x). \end{aligned}$$

b. This function is even because

$$\begin{aligned} h(-x) &= (-x)^2 + 1 \\ &= x^2 + 1 \\ &= h(x). \end{aligned}$$

c. Substituting $-x$ for x produces

$$\begin{aligned} f(-x) &= (-x)^3 - 1 \\ &= -x^3 - 1. \end{aligned}$$

Because $f(x) = x^3 - 1$ and $-f(x) = -x^3 + 1$, you can conclude that $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. So, the function is neither even nor odd.

CHECKPOINT Now try Exercise 61.

Graphical Solution

a. In Figure 1.37, the graph is symmetric with respect to the origin. So, this function is odd.

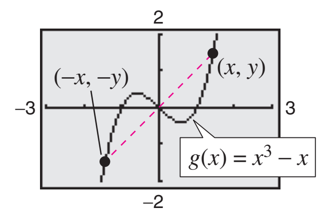


Figure 1.37

b. In Figure 1.38, the graph is symmetric with respect to the y-axis. So, this function is even.

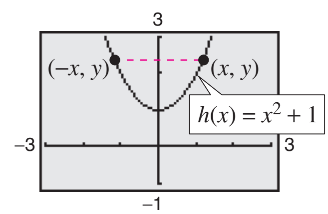


Figure 1.38

c. In Figure 1.39, the graph is neither symmetric with respect to the origin nor with respect to the y-axis. So, this function is neither even nor odd.

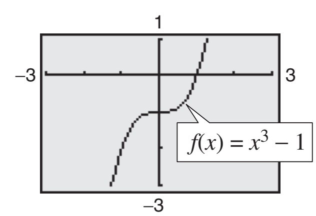


Figure 1.39

To help visualize symmetry with respect to the origin, place a pin at the origin of a graph and rotate the graph 180°. If the result after rotation coincides with the original graph, the graph is symmetric with respect to the origin.

1.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

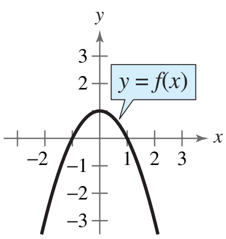
Vocabulary Check

Fill in the blanks.

1. The graph of a function f is a collection of _______ (x, y) such that x is in the domain of f.
2. The _______ is used to determine whether the graph of an equation is a function of y in terms of x.
3. A function f is _______ on an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.
4. A function value $f(a)$ is a relative _______ of f if there exists an interval (x_1, x_2) containing a such that $x_1 < x < x_2$ implies $f(a) \leq f(x)$.
5. The function $f(x) = [\![x]\!]$ is called the _______ function, and is an example of a step function.
6. A function f is _______ if, for each x in the domain of f, $f(-x) = f(x)$.

In Exercises 1–4, use the graph of the function to find the domain and range of f. Then find $f(0)$.

1.



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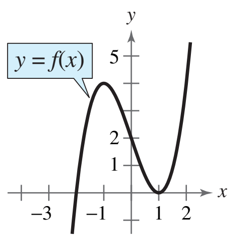

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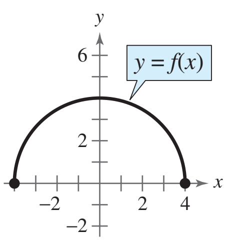

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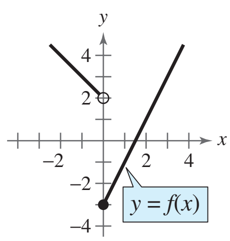


In Exercises 5–10, use a graphing utility to graph the function and estimate its domain and range. Then find the domain and range algebraically.

5. $f(x) = 2x^2 + 3$
6. $f(x) = -x^2 - 1$
7. $f(x) = \sqrt{x - 1}$
8. $h(t) = \sqrt{4 - t^2}$
9. $f(x) = |x + 3|$
10. $f(x) = -\frac{1}{4}|x - 5|$

In Exercises 11–14, use the given function to answer the questions.

(a) Determine the domain of the function.

(b) Find the value(s) of x such that $f(x) = 0$.

(c) The values of x from part (b) are referred to as what graphically?

(d) Find $f(0)$, if possible.

(e) The value from part (d) is referred to as what graphically?

(f) What is the value of f at $x = 1$? What are the coordinates of the point?

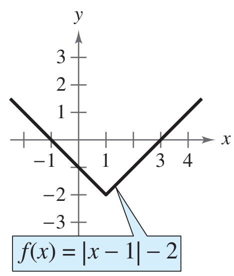
(g) What is the value of f at $x = -1$? What are the coordinates of the point?

(h) The coordinates of the point on the graph of f at which $x = -3$. can be labeled $(-3, f(-3))$ or $(-3,$ ______ $)$.

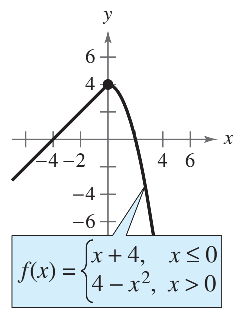
11. $f(x) = x^2 - x - 6$

12. $f(x) = x^3 - 4x$

13.

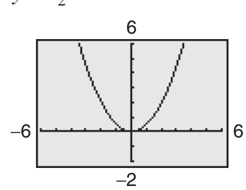


14.

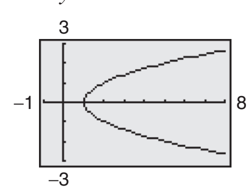


In Exercises 15–18, use the Vertical Line Test to determine whether y is a function of x. Describe how you can use a graphing utility to produce the given graph.

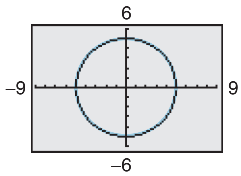
15. $y = \frac{1}{2}x^2$



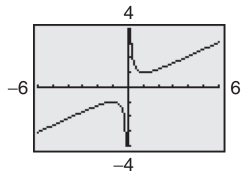
16. $x - y^2 = 1$



17. $x^2 + y^2 = 25$

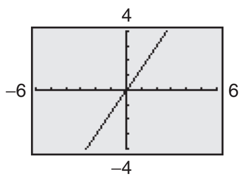


18. $x^2 = 2xy - 1$

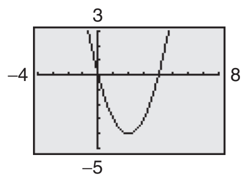


In Exercises 19–22, determine the open intervals over which the function is increasing, decreasing, or constant.

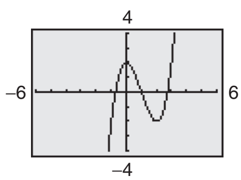
19. $f(x) = \frac{3}{2}x$



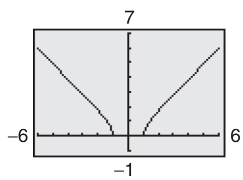
20. $f(x) = x^2 - 4x$



21. $f(x) = x^3 - 3x^2 + 2$



22. $f(x) = \sqrt{x^2 - 1}$



In Exercises 23–30, (a) use a graphing utility to graph the function and (b) determine the open intervals on which the function is increasing, decreasing, or constant.

23. $f(x) = 3$

24. $f(x) = x$

25. $f(x) = x^{2/3}$

26. $f(x) = -x^{3/4}$

27. $f(x) = x\sqrt{x + 3}$

28. $f(x) = \sqrt{1 - x}$

29. $f(x) = |x + 1| + |x - 1|$

30. $f(x) = -|x + 4| - |x + 1|$

In Exercises 31–36, use a graphing utility to approximate any relative minimum or relative maximum values of the function.

31. $f(x) = x^2 - 6x$

32. $f(x) = 3x^2 - 2x - 5$

33. $y = 2x^3 + 3x^2 - 12x$

34. $y = x^3 - 6x^2 + 15$

35. $h(x) = (x - 1)\sqrt{x}$

36. $g(x) = x\sqrt{4 - x}$

In Exercises 37–42, (a) approximate the relative minimum or relative maximum values of the function by sketching its graph using the point-plotting method, (b) use a graphing utility to approximate any relative minimum or relative maximum values, and (c) compare your answers from parts (a) and (b).

37. $f(x) = x^2 - 4x - 5$

38. $f(x) = 3x^2 - 12x$

39. $f(x) = x^3 - 3x$

40. $f(x) = -x^3 + 3x^2$

41. $f(x) = 3x^2 - 6x + 1$

42. $f(x) = 8x - 4x^2$

In Exercises 43–50, sketch the graph of the piecewise-defined function by hand.

43. $f(x) = \begin{cases} 2x + 3, & x < 0 \\ 3 - x, & x \geq 0 \end{cases}$

44. $f(x) = \begin{cases} x + 6, & x \leq -4 \\ 2x - 4, & x > -4 \end{cases}$

45. $f(x) = \begin{cases} \sqrt{4 + x}, & x < 0 \\ \sqrt{4 - x}, & x \geq 0 \end{cases}$

46. $f(x) = \begin{cases} 1 - (x - 1)^2, & x \leq 2 \\ \sqrt{x - 2}, & x > 2 \end{cases}$

47. $f(x) = \begin{cases} x + 3, & x \leq 0 \\ 3, & 0 < x \leq 2 \\ 2x - 1, & x > 2 \end{cases}$

48. $g(x) = \begin{cases} x + 5, & x \leq -3 \\ -2, & -3 < x < 1 \\ 5x - 4, & x \geq 1 \end{cases}$

49. $f(x) = \begin{cases} 2x + 1, & x \leq -1 \\ x^2 - 2, & x > -1 \end{cases}$

50. $h(x) = \begin{cases} 3 + x, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

Library of Parent Functions In Exercises 51–56, sketch the graph of the function by hand. Then use a graphing utility to verify the graph.

51. $f(x) = [\![x]\!] + 2$

52. $f(x) = [\![x]\!] - 3$

53. $f(x) = [\![x - 1]\!] + 2$

54. $f(x) = [\![x - 2]\!] + 1$

55. $f(x) = [\![2x]\!]$

56. $f(x) = [\![4x]\!]$

In Exercises 57 and 58, use a graphing utility to graph the function. State the domain and range of the function. Describe the pattern of the graph.

57. $s(x) = 2\left(\frac{1}{4}x - [\![\frac{1}{4}x]\!]\right)$

58. $g(x) = 2\left(\frac{1}{4}x - [\![\frac{1}{4}x]\!]\right)^2$

In Exercises 59–66, algebraically determine whether the function is even, odd, or neither. Verify your answer using a graphing utility.

59. $f(t) = t^2 + 2t - 3$
60. $f(x) = x^6 - 2x^2 + 3$
61. $g(x) = x^3 - 5x$
62. $h(x) = x^3 - 5$
63. $f(x) = x\sqrt{1 - x^2}$
64. $f(x) = x\sqrt{x + 5}$
65. $g(s) = 4s^{2/3}$
66. $f(s) = 4s^{3/2}$

Think About It **In Exercises 67–72, find the coordinates of a second point on the graph of a function f if the given point is on the graph and the function is (a) even and (b) odd.**

67. $\left(-\frac{3}{2}, 4\right)$
68. $\left(-\frac{5}{3}, -7\right)$
69. $(4, 9)$
70. $(5, -1)$
71. $(x, -y)$
72. $(2a, 2c)$

In Exercises 73–82, use a graphing utility to graph the function and determine whether it is even, odd, or neither. Verify your answer algebraically.

73. $f(x) = 5$
74. $f(x) = -9$
75. $f(x) = 3x - 2$
76. $f(x) = 5 - 3x$
77. $h(x) = x^2 - 4$
78. $f(x) = -x^2 - 8$
79. $f(x) = \sqrt{1 - x}$
80. $g(t) = \sqrt[3]{t - 1}$
81. $f(x) = |x + 2|$
82. $f(x) = -|x - 5|$

In Exercises 83–86, graph the function and determine the interval(s) (if any) on the real axis for which $f(x) \geq 0$. Use a graphing utility to verify your results.

83. $f(x) = 4 - x$
84. $f(x) = 4x + 2$
85. $f(x) = x^2 - 9$
86. $f(x) = x^2 - 4x$

87. ***Communications*** The cost of using a telephone calling card is \$1.05 for the first minute and \$0.38 for each additional minute or portion of a minute.

(a) A customer needs a model for the cost C of using the calling card for a call lasting t minutes. Which of the following is the appropriate model?

$$C_1(t) = 1.05 + 0.38[\![t - 1]\!]$$

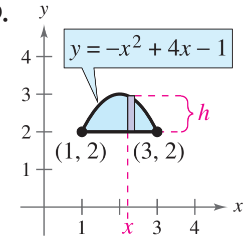
$$C_2(t) = 1.05 - 0.38[\![-(t - 1)]\!]$$

(b) Use a graphing utility to graph the appropriate model. Use the *value* feature or the *zoom* and *trace* features to estimate the cost of a call lasting 18 minutes and 45 seconds.

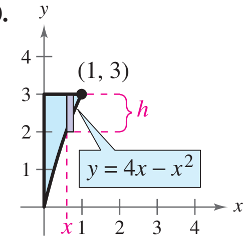
88. ***Delivery Charges*** The cost of sending an overnight package from New York to Atlanta is \$9.80 for a package weighing up to but not including 1 pound and \$2.50 for each additional pound or portion of a pound. Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds, where $x > 0$. Sketch the graph of the function.

In Exercises 89 and 90, write the height h of the rectangle as a function of x.

89.



90.



91. ***Population*** During a 14 year period from 1990 to 2004, the population P (in thousands) of West Virginia fluctuated according to the model

$$P = 0.0108t^4 - 0.211t^3 + 0.40t^2 + 7.9t + 1791,$$
$$0 \leq t \leq 14$$

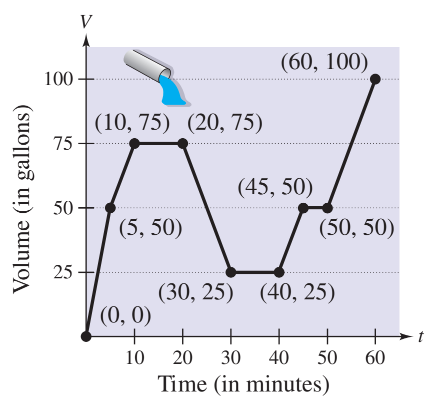
where t represents the year, with $t = 0$ corresponding to 1990. (Source: U.S. Census Bureau)

(a) Use a graphing utility to graph the model over the appropriate domain.

(b) Use the graph from part (a) to determine during which years the population was increasing. During which years was the population decreasing?

(c) Approximate the maximum population between 1990 and 2004.

92. ***Fluid Flow*** The intake pipe of a 100-gallon tank has a flow rate of 10 gallons per minute, and two drain pipes have a flow rate of 5 gallons per minute each. The graph shows the volume V of fluid in the tank as a function of time t. Determine in which pipes the fluid is flowing in specific subintervals of the one-hour interval of time shown on the graph. (There are many correct answers.)



Synthesis

True or False? **In Exercises 93 and 94, determine whether the statement is true or false. Justify your answer.**

93. A function with a square root cannot have a domain that is the set of all real numbers.

94. It is possible for an odd function to have the interval $[0, \infty)$ as its domain.

Think About It **In Exercises 95–100, match the graph of the function with the best choice that describes the situation.**

(a) The air temperature at a beach on a sunny day

(b) The height of a football kicked in a field goal attempt

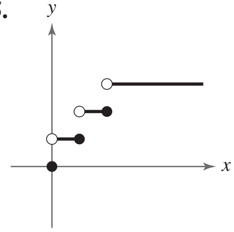
(c) The number of children in a family over time

(d) The population of California as a function of time

(e) The depth of the tide at a beach over a 24-hour period

(f) The number of cupcakes on a tray at a party

95.



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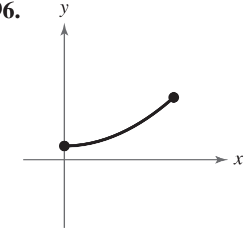

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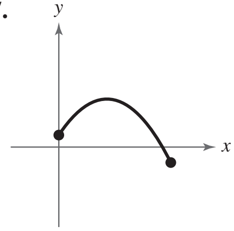

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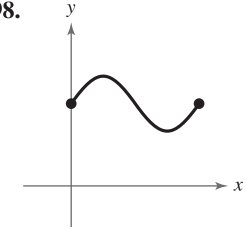

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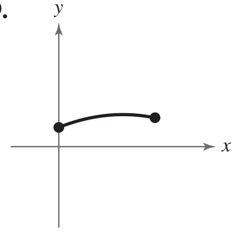
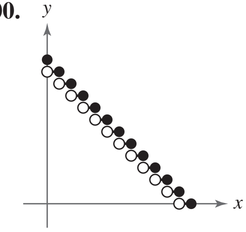


100.



101. ***Proof*** Prove that a function of the following form is odd.

$$y = a_{2n+1}x^{2n+1} + a_{2n-1}x^{2n-1} + \cdot \cdot \cdot + a_3x^3 + a_1x$$

102. ***Proof*** Prove that a function of the following form is even.

$$y = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdot \cdot \cdot + a_2x^2 + a_0$$

103. If f is an even function, determine if g is even, odd, or neither. Explain.

(a) $g(x) = -f(x)$ (b) $g(x) = f(-x)$

(c) $g(x) = f(x) - 2$ (d) $g(x) = -f(x - 2)$

104. ***Think About It*** Does the graph in Exercise 16 represent x as a function of y? Explain.

105. ***Think About It*** Does the graph in Exercise 17 represent x as a function of y? Explain.

106. ***Writing*** Write a short paragraph describing three different functions that represent the behaviors of quantities between 1995 and 2006. Describe one quantity that decreased during this time, one that increased, and one that was constant. Present your results graphically.

Skills Review

In Exercises 107–110, identify the terms. Then identify the coefficients of the variable terms of the expression.

107. $-2x^2 + 8x$ **108.** $10 + 3x$

109. $\dfrac{x}{3} - 5x^2 + x^3$ **110.** $7x^4 + \sqrt{2}x^2$

In Exercises 111–114, find (a) the distance between the two points and (b) the midpoint of the line segment joining the points.

111. $(-2, 7), (6, 3)$

112. $(-5, 0), (3, 6)$

113. $\left(\frac{5}{2}, -1\right), \left(-\frac{3}{2}, 4\right)$

114. $\left(-6, \frac{2}{3}\right), \left(\frac{3}{4}, \frac{1}{6}\right)$

In Exercises 115–118, evaluate the function at each specified value of the independent variable and simplify.

115. $f(x) = 5x - 1$

(a) $f(6)$ (b) $f(-1)$ (c) $f(x - 3)$

116. $f(x) = -x^2 - x + 3$

(a) $f(4)$ (b) $f(-2)$ (c) $f(x - 2)$

117. $f(x) = x\sqrt{x - 3}$

(a) $f(3)$ (b) $f(12)$ (c) $f(6)$

118. $f(x) = -\frac{1}{2}x|x + 1|$

(a) $f(-4)$ (b) $f(10)$ (c) $f\left(-\frac{2}{3}\right)$

In Exercises 119 and 120, find the difference quotient and simplify your answer.

119. $f(x) = x^2 - 2x + 9, \dfrac{f(3 + h) - f(3)}{h}, h \neq 0$

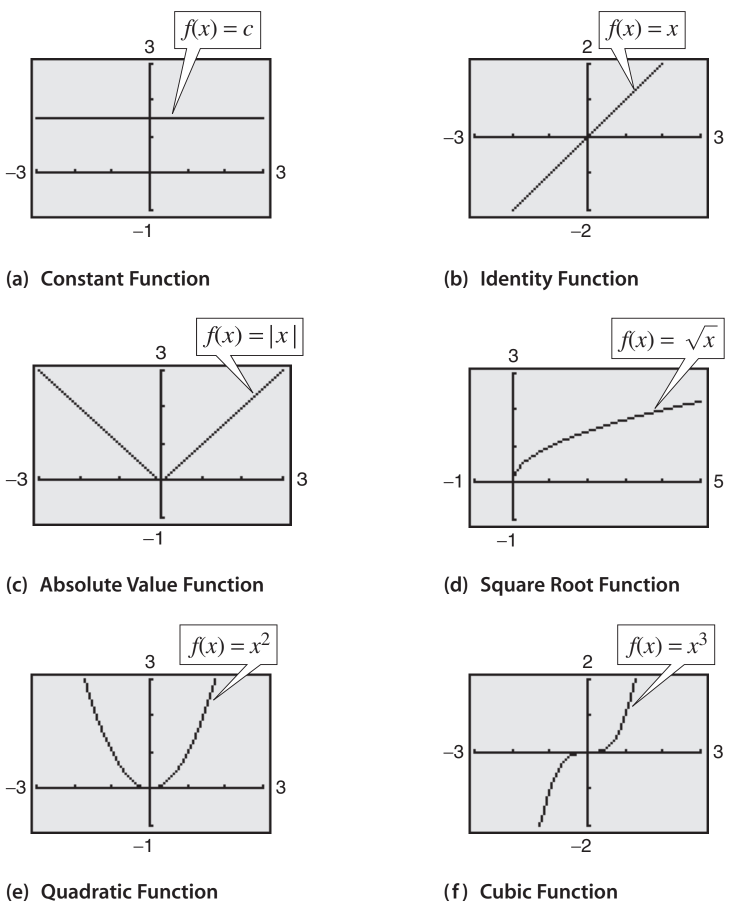
120. $f(x) = 5 + 6x - x^2, \dfrac{f(6 + h) - f(6)}{h}, h \neq 0$

1.4 Shifting, Reflecting, and Stretching Graphs

Summary of Graphs of Parent Functions

One of the goals of this text is to enable you to build your intuition for the basic shapes of the graphs of different types of functions. For instance, from your study of lines in Section 1.1, you can determine the basic shape of the graph of the linear function $f(x) = mx + b$. Specifically, you know that the graph of this function is a line whose slope is m and whose y-intercept is $(0, b)$.

The six graphs shown in Figure 1.40 represent the most commonly used functions in algebra. Familiarity with the basic characteristics of these simple graphs will help you analyze the shapes of more complicated graphs.



(a) Constant Function

(b) Identity Function

(c) Absolute Value Function

(d) Square Root Function

(e) Quadratic Function

(f) Cubic Function

Figure 1.40

Throughout this section, you will discover how many complicated graphs are derived by shifting, stretching, shrinking, or reflecting the parent graphs shown above. Shifts, stretches, shrinks, and reflections are called *transformations*. Many graphs of functions can be created from combinations of these transformations.

What you should learn

- Recognize graphs of parent functions.
- Use vertical and horizontal shifts and reflections to graph functions.
- Use nonrigid transformations to graph functions.

Why you should learn it

Recognizing the graphs of parent functions and knowing how to shift, reflect, and stretch graphs of functions can help you sketch a wide variety of simple functions by hand. This skill is useful in sketching graphs of functions that model real-life data. For example, in Exercise 57 on page 49, you are asked to sketch a function that models the amount of fuel used by vans, pickups, and sport utility vehicles from 1990 through 2003.

Tim Boyle/Getty Images

Vertical and Horizontal Shifts

Many functions have graphs that are simple transformations of the graphs of parent functions summarized in Figure 1.40. For example, you can obtain the graph of

$$h(x) = x^2 + 2$$

by shifting the graph of $f(x) = x^2$ two units *upward,* as shown in Figure 1.41. In function notation, h and f are related as follows.

$$\begin{aligned} h(x) &= x^2 + 2 \\ &= f(x) + 2 \qquad \text{Upward shift of two units} \end{aligned}$$

Similarly, you can obtain the graph of

$$g(x) = (x - 2)^2$$

by shifting the graph of $f(x) = x^2$ two units to the *right,* as shown in Figure 1.42. In this case, the functions g and f have the following relationship.

$$\begin{aligned} g(x) &= (x - 2)^2 \\ &= f(x - 2) \qquad \text{Right shift of two units} \end{aligned}$$

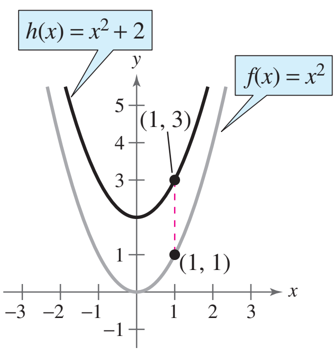


Figure 1.41 ***Vertical shift upward: two units***

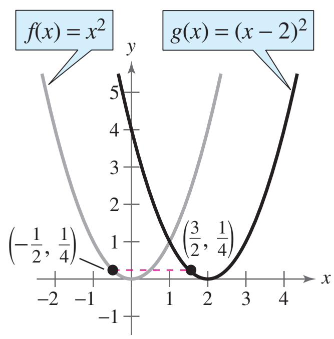


Figure 1.42 ***Horizontal shift to the right: two units***

Exploration

Use a graphing utility to display (in the same viewing window) the graphs of $y = x^2 + c$, where $c = -2, 0, 2,$ and 4. Use the results to describe the effect that c has on the graph.

Use a graphing utility to display (in the same viewing window) the graphs of $y = (x + c)^2$, where $c = -2, 0, 2,$ and 4. Use the results to describe the effect that c has on the graph.

The following list summarizes vertical and horizontal shifts.

Vertical and Horizontal Shifts

Let c be a positive real number. **Vertical and horizontal shifts** in the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units *upward*: $\quad h(x) = f(x) + c$
2. Vertical shift c units *downward*: $\quad h(x) = f(x) - c$
3. Horizontal shift c units to the *right*: $\quad h(x) = f(x - c)$
4. Horizontal shift c units to the *left*: $\quad h(x) = f(x + c)$

In items 3 and 4, be sure you see that $h(x) = f(x - c)$ corresponds to a *right* shift and $h(x) = f(x + c)$ corresponds to a *left* shift for $c > 0$.

Example 1 Shifts in the Graph of a Function

Compare the graph of each function with the graph of $f(x) = x^3$.

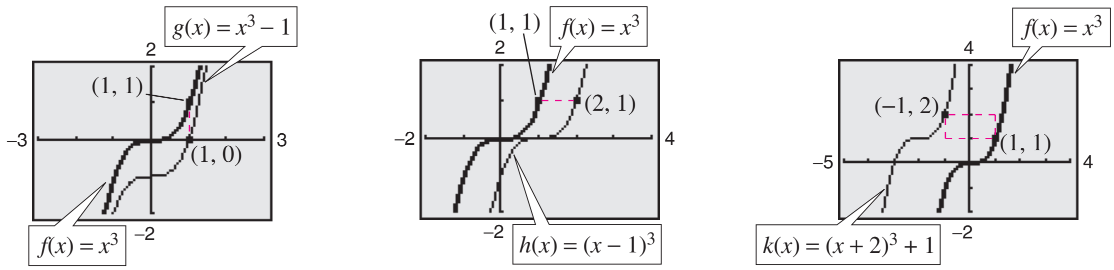
a. $g(x) = x^3 - 1$ **b.** $h(x) = (x - 1)^3$ **c.** $k(x) = (x + 2)^3 + 1$

Solution

a. Graph $f(x) = x^3$ and $g(x) = x^3 - 1$ [see Figure 1.43(a)]. You can obtain the graph of g by shifting the graph of f one unit downward.

b. Graph $f(x) = x^3$ and $h(x) = (x - 1)^3$ [see Figure 1.43(b)]. You can obtain the graph of h by shifting the graph of f one unit to the right.

c. Graph $f(x) = x^3$ and $k(x) = (x + 2)^3 + 1$ [see Figure 1.43(c)]. You can obtain the graph of k by shifting the graph of f two units to the left and then one unit upward.



(a) Vertical shift: one unit downward **(b)** Horizontal shift: one unit right **(c)** Two units left and one unit upward

Figure 1.43

CHECKPOINT Now try Exercise 23.

Example 2 Finding Equations from Graphs

The graph of $f(x) = x^2$ is shown in Figure 1.44. Each of the graphs in Figure 1.45 is a transformation of the graph of f. Find an equation for each function.

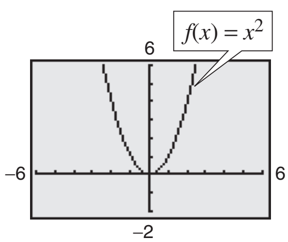
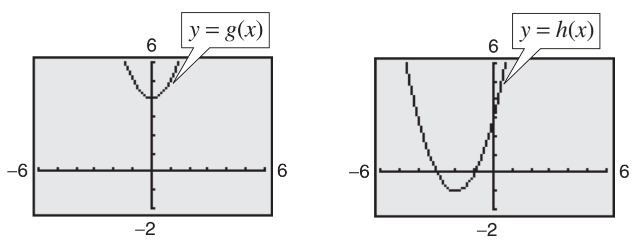


Figure 1.44



(a) **(b)**

Figure 1.45

Solution

a. The graph of g is a vertical shift of four units upward of the graph of $f(x) = x^2$. So, the equation for g is $g(x) = x^2 + 4$.

b. The graph of h is a horizontal shift of two units to the left, and a vertical shift of one unit downward, of the graph of $f(x) = x^2$. So, the equation for h is $h(x) = (x + 2)^2 - 1$.

CHECKPOINT Now try Exercise 17.

Reflecting Graphs

Another common type of transformation is called a **reflection.** For instance, if you consider the x-axis to be a mirror, the graph of $h(x) = -x^2$ is the mirror image (or reflection) of the graph of $f(x) = x^2$ (see Figure 1.46).

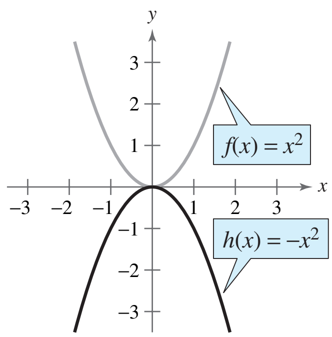


Figure 1.46

> ### Exploration
> Compare the graph of each function with the graph of $f(x) = x^2$ by using a graphing utility to graph the function and f in the same viewing window. Describe the transformation.
>
> **a.** $g(x) = -x^2$
>
> **b.** $h(x) = (-x)^2$

> **Reflections in the Coordinate Axes**
>
> Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows.
>
> **1.** Reflection in the x-axis: $h(x) = -f(x)$
>
> **2.** Reflection in the y-axis: $h(x) = f(-x)$

Example 3 Finding Equations from Graphs

The graph of $f(x) = x^4$ is shown in Figure 1.47. Each of the graphs in Figure 1.48 is a transformation of the graph of f. Find an equation for each function.

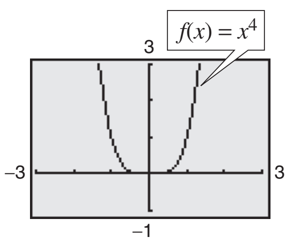
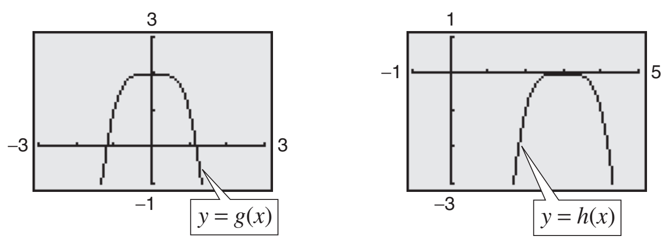


Figure 1.47



(a) **(b)**

Figure 1.48

Solution

a. The graph of g is a reflection in the x-axis *followed by* an upward shift of two units of the graph of $f(x) = x^4$. So, the equation for g is $g(x) = -x^4 + 2$.

b. The graph of h is a horizontal shift of three units to the right *followed by* a reflection in the x-axis of the graph of $f(x) = x^4$. So, the equation for h is $h(x) = -(x - 3)^4$.

CHECKPOINT Now try Exercise 19.

Example 4 Reflections and Shifts

Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

a. $g(x) = -\sqrt{x}$ **b.** $h(x) = \sqrt{-x}$ **c.** $k(x) = -\sqrt{x+2}$

Algebraic Solution

a. Relative to the graph of $f(x) = \sqrt{x}$, the graph of g is a reflection in the x-axis because

$$g(x) = -\sqrt{x}$$
$$= -f(x).$$

b. The graph of h is a reflection of the graph of $f(x) = \sqrt{x}$ in the y-axis because

$$h(x) = \sqrt{-x}$$
$$= f(-x).$$

c. From the equation

$$k(x) = -\sqrt{x+2}$$
$$= -f(x+2)$$

you can conclude that the graph of k is a left shift of two units, followed by a reflection in the x-axis, of the graph of $f(x) = \sqrt{x}$.

CHECKPOINT Now try Exercise 21.

Graphical Solution

a. Use a graphing utility to graph f and g in the same viewing window. From the graph in Figure 1.49, you can see that the graph of g is a reflection of the graph of f in the x-axis.

b. Use a graphing utility to graph f and h in the same viewing window. From the graph in Figure 1.50, you can see that the graph of h is a reflection of the graph of f in the y-axis.

c. Use a graphing utility to graph f and k in the same viewing window. From the graph in Figure 1.51, you can see that the graph of k is a left shift of two units of the graph of f, followed by a reflection in the x-axis.

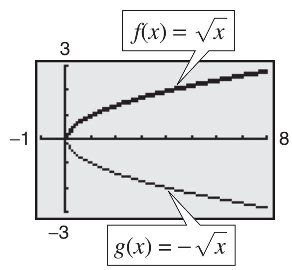


Figure 1.49

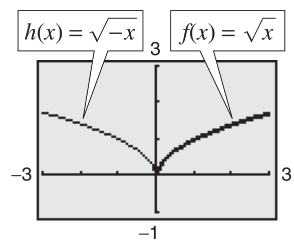


Figure 1.50

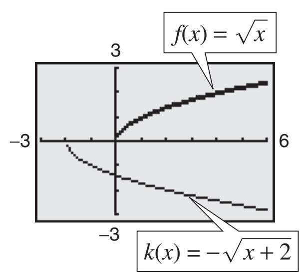


Figure 1.51

When graphing functions involving square roots, remember that the domain must be restricted to exclude negative numbers inside the radical. For instance, here are the domains of the functions in Example 4.

Domain of $g(x) = -\sqrt{x}$: $x \ge 0$

Domain of $h(x) = \sqrt{-x}$: $x \le 0$

Domain of $k(x) = -\sqrt{x+2}$: $x \ge -2$

Nonrigid Transformations

Horizontal shifts, vertical shifts, and reflections are called **rigid transformations** because the basic shape of the graph is unchanged. These transformations change only the *position* of the graph in the coordinate plane. **Nonrigid transformations** are those that cause a *distortion*—a change in the shape of the original graph. For instance, a nonrigid transformation of the graph of $y = f(x)$ is represented by $y = cf(x)$, where the transformation is a **vertical stretch** if $c > 1$ and a **vertical shrink** if $0 < c < 1$. Another nonrigid transformation of the graph of $y = f(x)$ is represented by $h(x) = f(cx)$, where the transformation is a **horizontal shrink** if $c > 1$ and a **horizontal stretch** if $0 < c < 1$.

Example 5 Nonrigid Transformations

Compare the graph of each function with the graph of $f(x) = |x|$.

a. $h(x) = 3|x|$

b. $g(x) = \frac{1}{3}|x|$

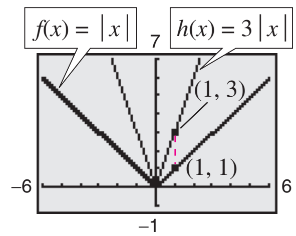


Figure 1.52

Solution

a. Relative to the graph of $f(x) = |x|$, the graph of

$$h(x) = 3|x| = 3f(x)$$

is a vertical stretch (each y-value is multiplied by 3) of the graph of f. (See Figure 1.52.)

b. Similarly, the graph of

$$g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$$

is a vertical shrink $\left(\text{each } y\text{-value is multiplied by } \frac{1}{3}\right)$ of the graph of f. (See Figure 1.53.)

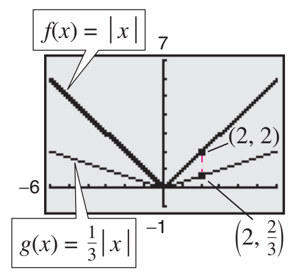


Figure 1.53

CHECKPOINT Now try Exercise 31.

Example 6 Nonrigid Transformations

Compare the graph of $h(x) = f\left(\frac{1}{2}x\right)$ with the graph of $f(x) = 2 - x^3$.

Solution

Relative to the graph of $f(x) = 2 - x^3$, the graph of

$$h(x) = f\left(\frac{1}{2}x\right) = 2 - \left(\frac{1}{2}x\right)^3 = 2 - \frac{1}{8}x^3$$

is a horizontal stretch (each x-value is multiplied by 2) of the graph of f. (See Figure 1.54.)

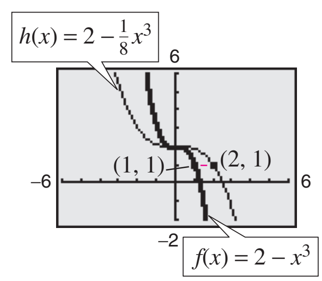


Figure 1.54

CHECKPOINT Now try Exercise 39.

1.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

In Exercises 1–5, fill in the blanks.

1. The graph of a ________ is U-shaped.

2. The graph of an ________ is V-shaped.

3. Horizontal shifts, vertical shifts, and reflections are called ________ .

4. A reflection in the x-axis of $y = f(x)$ is represented by $h(x) =$ ________ , while a reflection in the y-axis of $y = f(x)$ is represented by $h(x) =$ ________ .

5. A nonrigid transformation of $y = f(x)$ represented by $cf(x)$ is a vertical stretch if ________ and a vertical shrink if ________ .

6. Match the rigid transformation of $y = f(x)$ with the correct representation, where $c > 0$.

(a) $h(x) = f(x) + c$ (i) horizontal shift c units to the left

(b) $h(x) = f(x) - c$ (ii) vertical shift c units upward

(c) $h(x) = f(x - c)$ (iii) horizontal shift c units to the right

(d) $h(x) = f(x + c)$ (iv) vertical shift c units downward

In Exercises 1–12, sketch the graphs of the three functions by hand on the same rectangular coordinate system. Verify your result with a graphing utility.

1. $f(x) = x$
$g(x) = x - 4$
$h(x) = 3x$

2. $f(x) = \frac{1}{2}x$
$g(x) = \frac{1}{2}x + 2$
$h(x) = \frac{1}{2}(x - 2)$

3. $f(x) = x^2$
$g(x) = x^2 + 2$
$h(x) = (x - 2)^2$

4. $f(x) = x^2$
$g(x) = x^2 - 4$
$h(x) = (x + 2)^2 + 1$

5. $f(x) = -x^2$
$g(x) = -x^2 + 1$
$h(x) = -(x - 2)^2$

6. $f(x) = (x - 2)^2$
$g(x) = (x + 2)^2 + 2$
$h(x) = -(x - 2)^2 - 1$

7. $f(x) = x^2$
$g(x) = \frac{1}{2}x^2$
$h(x) = (2x)^2$

8. $f(x) = x^2$
$g(x) = \frac{1}{4}x^2 + 2$
$h(x) = -\frac{1}{4}x^2$

9. $f(x) = |x|$
$g(x) = |x| - 1$
$h(x) = |x - 3|$

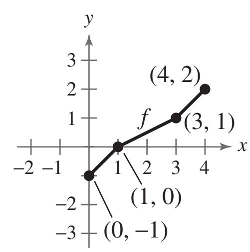
10. $f(x) = |x|$
$g(x) = |x + 3|$
$h(x) = -2|x + 2| - 1$

11. $f(x) = \sqrt{x}$
$g(x) = \sqrt{x + 1}$
$h(x) = \sqrt{x - 2} + 1$

12. $f(x) = \sqrt{x}$
$g(x) = \frac{1}{2}\sqrt{x}$
$h(x) = -\sqrt{x + 4}$

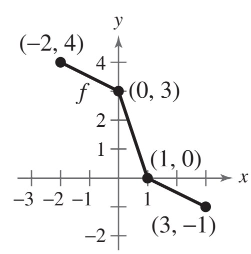
13. Use the graph of f to sketch each graph. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

(a) $y = f(x) + 2$
(b) $y = -f(x)$
(c) $y = f(x - 2)$
(d) $y = f(x + 3)$
(e) $y = 2f(x)$
(f) $y = f(-x)$
(g) $y = f\left(\frac{1}{2}x\right)$



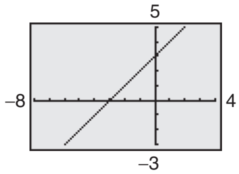
14. Use the graph of f to sketch each graph. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) $y = f(x) - 1$
(b) $y = f(x + 1)$
(c) $y = f(x - 1)$
(d) $y = -f(x - 2)$
(e) $y = f(-x)$
(f) $y = \frac{1}{2}f(x)$
(g) $y = f(2x)$

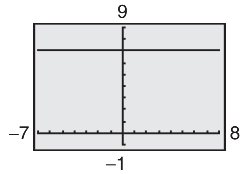


In Exercises 15–20, identify the parent function and describe the transformation shown in the graph. Write an equation for the graphed function.

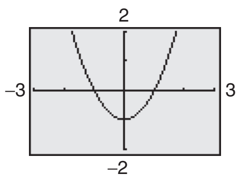
15.



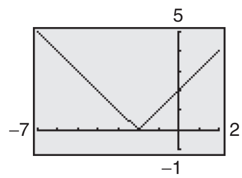
16.



17.



18.



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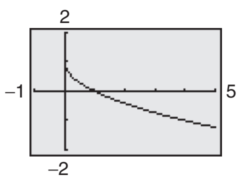
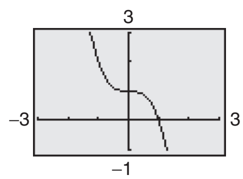


20.



In Exercises 21–26, compare the graph of the function with the graph of $f(x) = \sqrt{x}$.

21. $y = -\sqrt{x} - 1$ **22.** $y = \sqrt{x} + 2$

23. $y = \sqrt{x - 2}$ **24.** $y = \sqrt{x + 4}$

25. $y = 2\sqrt{x}$ **26.** $y = \sqrt{-x + 3}$

In Exercises 27–32, compare the graph of the function with the graph of $f(x) = |x|$.

27. $y = |x + 5|$ **28.** $y = |x| - 3$

29. $y = -|x|$ **30.** $y = |-x|$

31. $y = 4|x|$ **32.** $y = \left|\frac{1}{2}x\right|$

In Exercises 33–38, compare the graph of the function with the graph of $f(x) = x^3$.

33. $g(x) = 4 - x^3$ **34.** $g(x) = -(x - 1)^3$

35. $h(x) = \frac{1}{4}(x + 2)^3$ **36.** $h(x) = -2(x - 1)^3 + 3$

37. $p(x) = \left(\frac{1}{3}x\right)^3 + 2$ **38.** $p(x) = [3(x - 2)]^3$

In Exercises 39–42, use a graphing utility to graph the three functions in the same viewing window. Describe the graphs of g and h relative to the graph of f.

39. $f(x) = x^3 - 3x^2$
$g(x) = f(x + 2)$
$h(x) = \frac{1}{2}f(x)$

40. $f(x) = x^3 - 3x^2 + 2$
$g(x) = f(x - 1)$
$h(x) = f(3x)$

41. $f(x) = x^3 - 3x^2$
$g(x) = -\frac{1}{3}f(x)$
$h(x) = f(-x)$

42. $f(x) = x^3 - 3x^2 + 2$
$g(x) = -f(x)$
$h(x) = f(2x)$

In Exercises 43–56, g is related to one of the six parent functions on page 42. (a) Identify the parent function f. (b) Describe the sequence of transformations from f to g. (c) Sketch the graph of g by hand. (d) Use function notation to write g in terms of the parent function f.

43. $g(x) = 2 - (x + 5)^2$ **44.** $g(x) = -(x + 10)^2 + 5$

45. $g(x) = 3 + 2(x - 4)^2$ **46.** $g(x) = -\frac{1}{4}(x + 2)^2 - 2$

47. $g(x) = 3(x - 2)^3$ **48.** $g(x) = -\frac{1}{2}(x + 1)^3$

49. $g(x) = (x - 1)^3 + 2$ **50.** $g(x) = -(x + 3)^3 - 10$

51. $g(x) = |x + 4| + 8$ **52.** $g(x) = |x + 3| + 9$

53. $g(x) = -2|x - 1| - 4$ **54.** $g(x) = \frac{1}{2}|x - 2| - 3$

55. $g(x) = -\frac{1}{2}\sqrt{x + 3} - 1$ **56.** $g(x) = -\sqrt{x + 1} - 6$

57. ***Fuel Use*** The amounts of fuel F (in billions of gallons) used by vans, pickups, and SUVs (sport utility vehicles) from 1990 through 2003 are shown in the table. A model for the data can be approximated by the function $F(t) = 33.0 + 6.2\sqrt{t}$, where $t = 0$ represents 1990. (Source: U.S. Federal Highway Administration)

Year	Annual fuel use, F (in billions of gallons)
1990	35.6
1991	38.2
1992	40.9
1993	42.9
1994	44.1
1995	45.6
1996	47.4
1997	49.4
1998	50.5
1999	52.8
2000	52.9
2001	53.5
2002	55.2
2003	56.3

(a) Describe the transformation of the parent function $f(t) = \sqrt{t}$.

(b) Use a graphing utility to graph the model and the data in the same viewing window.

(c) Rewrite the function so that $t = 0$ represents 2003. Explain how you got your answer.

58. ***Finance*** The amounts M (in billions of dollars) of home mortgage debt outstanding in the United States from 1990 through 2004 can be approximated by the function

$$M(t) = 32.3t^2 + 3769$$

where $t = 0$ represents 1990. (Source: Board of Governors of the Federal Reserve System)

(a) Describe the transformation of the parent function $f(t) = t^2$.

(b) Use a graphing utility to graph the model over the interval $0 \le t \le 14$.

(c) According to the model, when will the amount of debt exceed 10 trillion dollars?

(d) Rewrite the function so that $t = 0$ represents 2000. Explain how you got your answer.

Synthesis

True or False? **In Exercises 59 and 60, determine whether the statement is true or false. Justify your answer.**

59. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

60. The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

61. ***Exploration*** Use the fact that the graph of $y = f(x)$ has x-intercepts at $x = 2$ and $x = -3$ to find the x-intercepts of the given graph. If not possible, state the reason.

(a) $y = f(-x)$ (b) $y = -f(x)$ (c) $y = 2f(x)$
(d) $y = f(x) + 2$ (e) $y = f(x - 3)$

62. ***Exploration*** Use the fact that the graph of $y = f(x)$ has x-intercepts at $x = -1$ and $x = 4$ to find the x-intercepts of the given graph. If not possible, state the reason.

(a) $y = f(-x)$ (b) $y = -f(x)$ (c) $y = 2f(x)$
(d) $y = f(x) - 1$ (e) $y = f(x - 2)$

63. ***Exploration*** Use the fact that the graph of $y = f(x)$ is increasing on the interval $(-\infty, 2)$ and decreasing on the interval $(2, \infty)$ to find the intervals on which the graph is increasing and decreasing. If not possible, state the reason.

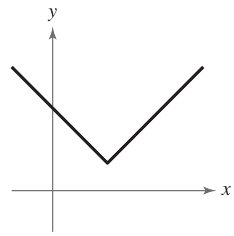
(a) $y = f(-x)$ (b) $y = -f(x)$ (c) $y = 2f(x)$
(d) $y = f(x) - 3$ (e) $y = f(x + 1)$

64. ***Exploration*** Use the fact that the graph of $y = f(x)$ is increasing on the intervals $(-\infty, -1)$ and $(2, \infty)$ and decreasing on the interval $(-1, 2)$ to find the intervals on which the graph is increasing and decreasing. If not possible, state the reason.

(a) $y = f(-x)$ (b) $y = -f(x)$ (c) $y = \frac{1}{2}f(x)$
(d) $y = -f(x - 1)$ (e) $y = f(x - 2) + 1$

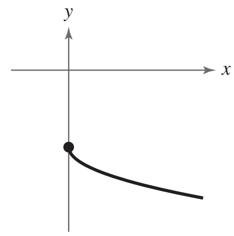
Library of Parent Functions **In Exercises 65–68, determine which equation(s) may be represented by the graph shown. There may be more than one correct answer.**

65.



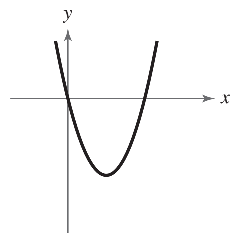
(a) $f(x) = |x + 2| + 1$
(b) $f(x) = |x - 1| + 2$
(c) $f(x) = |x - 2| + 1$
(d) $f(x) = 2 + |x - 2|$
(e) $f(x) = |(x - 2) + 1|$
(f) $f(x) = 1 - |x - 2|$

66.



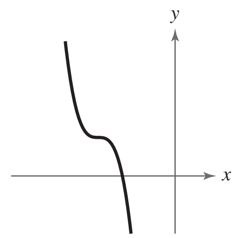
(a) $f(x) = -\sqrt{x} - 4$
(b) $f(x) = -4 - \sqrt{x}$
(c) $f(x) = -4 - \sqrt{-x}$
(d) $f(x) = \sqrt{-x} - 4$
(e) $f(x) = \sqrt{-x} + 4$
(f) $f(x) = \sqrt{x} - 4$

67.



(a) $f(x) = (x - 2)^2 - 2$
(b) $f(x) = (x + 4)^2 - 4$
(c) $f(x) = (x - 2)^2 - 4$
(d) $f(x) = (x + 2)^2 - 4$
(e) $f(x) = 4 - (x - 2)^2$
(f) $f(x) = 4 - (x + 2)^2$

68.



(a) $f(x) = -(x - 4)^3 + 2$
(b) $f(x) = -(x + 4)^3 + 2$
(c) $f(x) = -(x - 2)^3 + 4$
(d) $f(x) = (-x - 4)^3 + 2$
(e) $f(x) = (x + 4)^3 + 2$
(f) $f(x) = (-x + 4)^3 + 2$

Skills Review

In Exercises 69 and 70, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.

69. L_1: $(-2, -2), (2, 10)$
L_2: $(-1, 3), (3, 9)$

70. L_1: $(-1, -7), (4, 3)$
L_2: $(1, 5), (-2, -7)$

In Exercises 71–74, find the domain of the function.

71. $f(x) = \dfrac{4}{9 - x}$

72. $f(x) = \dfrac{\sqrt{x - 5}}{x - 7}$

73. $f(x) = \sqrt{100 - x^2}$

74. $f(x) = \sqrt[3]{16 - x^2}$

1.5 Combinations of Functions

What you should learn

- Add, subtract, multiply, and divide functions.
- Find compositions of one function with another function.
- Use combinations of functions to model and solve real-life problems.

Why you should learn it

Combining functions can sometimes help you better understand the big picture. For instance, Exercises 75 and 76 on page 60 illustrate how to use combinations of functions to analyze U.S. health care expenditures.

SuperStock

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two *functions* can be combined to create new functions. If $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, you can form the sum, difference, product, and quotient of f and g as follows.

$$\begin{aligned} f(x) + g(x) &= (2x - 3) + (x^2 - 1) \\ &= x^2 + 2x - 4 && \text{Sum} \\ f(x) - g(x) &= (2x - 3) - (x^2 - 1) \\ &= -x^2 + 2x - 2 && \text{Difference} \\ f(x) \cdot g(x) &= (2x - 3)(x^2 - 1) \\ &= 2x^3 - 3x^2 - 2x + 3 && \text{Product} \\ \frac{f(x)}{g(x)} &= \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1 && \text{Quotient} \end{aligned}$$

The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are common to the domains of f and g. In the case of the quotient $f(x)/g(x)$, there is the further restriction that $g(x) \neq 0$.

Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the sum, difference, product, and quotient of f and g are defined as follows.

1. Sum: $(f + g)(x) = f(x) + g(x)$
2. Difference: $(f - g)(x) = f(x) - g(x)$
3. Product: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

Example 1 Finding the Sum of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f + g)(x)$. Then evaluate the sum when $x = 2$.

Solution

$$(f + g)(x) = f(x) + g(x) = (2x + 1) + (x^2 + 2x - 1) = x^2 + 4x$$

When $x = 2$, the value of this sum is $(f + g)(2) = 2^2 + 4(2) = 12$.

CHECKPOINT Now try Exercise 7(a).

Example 2 Finding the Difference of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

Algebraic Solution

The difference of the functions f and g is

$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= (2x + 1) - (x^2 + 2x - 1)\\ &= -x^2 + 2.\end{aligned}$$

When $x = 2$, the value of this difference is

$$\begin{aligned}(f - g)(2) &= -(2)^2 + 2\\ &= -2.\end{aligned}$$

Note that $(f - g)(2)$ can also be evaluated as follows.

$$\begin{aligned}(f - g)(2) &= f(2) - g(2)\\ &= [2(2) + 1] - [2^2 + 2(2) - 1]\\ &= 5 - 7\\ &= -2\end{aligned}$$

CHECKPOINT Now try Exercise 7(b).

Graphical Solution

You can use a graphing utility to graph the difference of two functions. Enter the functions as follows (see Figure 1.55).

$$y_1 = 2x + 1$$

$$y_2 = x^2 + 2x - 1$$

$$y_3 = y_1 - y_2$$

Graph y_3 as shown in Figure 1.56. Then use the *value* feature or the *zoom* and *trace* features to estimate that the value of the difference when $x = 2$ is -2.

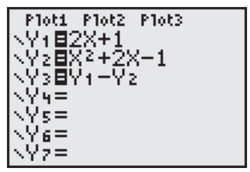


Figure 1.55

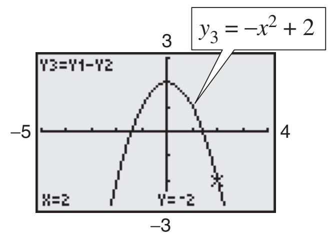


Figure 1.56

In Examples 1 and 2, both f and g have domains that consist of all real numbers. So, the domain of both $(f + g)$ and $(f - g)$ is also the set of all real numbers. Remember that any restrictions on the domains of f or g must be considered when forming the sum, difference, product, or quotient of f and g. For instance, the domain of $f(x) = 1/x$ is all $x \neq 0$, and the domain of $g(x) = \sqrt{x}$ is $[0, \infty)$. This implies that the domain of $(f + g)$ is $(0, \infty)$.

Example 3 Finding the Product of Two Functions

Given $f(x) = x^2$ and $g(x) = x - 3$, find $(fg)(x)$. Then evaluate the product when $x = 4$.

Solution

$$\begin{aligned}(fg)(x) &= f(x)g(x)\\ &= (x^2)(x - 3)\\ &= x^3 - 3x^2\end{aligned}$$

When $x = 4$, the value of this product is

$$(fg)(4) = 4^3 - 3(4)^2 = 16.$$

CHECKPOINT Now try Exercise 7(c).

Example 4 Finding the Quotient of Two Functions

Find $(f/g)(x)$ and $(g/f)(x)$ for the functions given by $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$. Then find the domains of f/g and g/f.

Solution

The quotient of f and g is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}},$$

and the quotient of g and f is

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4 - x^2}}{\sqrt{x}}.$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So, the domains for f/g and g/f are as follows.

Domain of $(f/g): [0, 2)$ Domain of $(g/f): (0, 2]$

CHECKPOINT Now try Exercise 7(d).

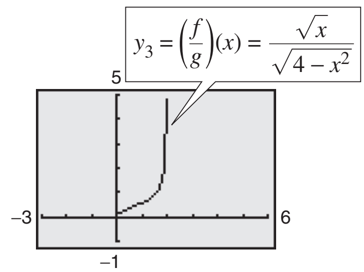


Figure 1.57

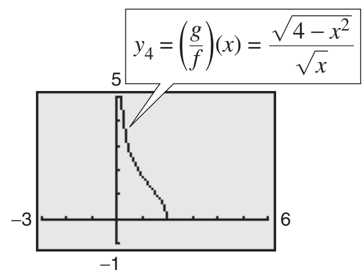


Figure 1.58

TECHNOLOGY TIP You can confirm the domain of f/g in Example 4 with your graphing utility by entering the three functions $y_1 = \sqrt{x}$, $y_2 = \sqrt{4 - x^2}$, and $y_3 = y_1/y_2$, and graphing y_3, as shown in Figure 1.57. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from 0 to 2 but do not include 2. So, you can estimate the domain of f/g to be $[0, 2)$. You can confirm the domain of g/f in Example 4 by entering $y_4 = y_2/y_1$ and graphing y_4, as shown in Figure 1.58. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from 0 to 2 but do not include 0. So, you can estimate the domain of g/f to be $(0, 2]$.

Compositions of Functions

Another way of combining two functions is to form the **composition** of one with the other. For instance, if $f(x) = x^2$ and $g(x) = x + 1$, the composition of f with g is

$$f(g(x)) = f(x + 1) = (x + 1)^2.$$

This composition is denoted as $f \circ g$ and is read as "f composed with g."

Definition of Composition of Two Functions

The **composition** of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 1.59.)

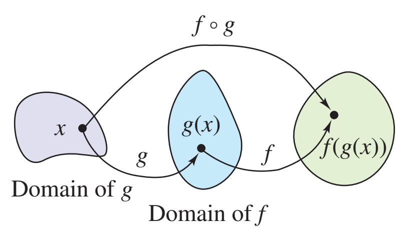


Figure 1.59

Example 5 Forming the Composition of f with g

Find $(f \circ g)(x)$ for $f(x) = \sqrt{x}$, $x \geq 0$, and $g(x) = x - 1$, $x \geq 1$. If possible, find $(f \circ g)(2)$ and $(f \circ g)(0)$.

Solution

$$(f \circ g)(x) = f(g(x)) \qquad \text{Definition of } f \circ g$$
$$= f(x - 1) \qquad \text{Definition of } g(x)$$
$$= \sqrt{x - 1}, \quad x \geq 1 \qquad \text{Definition of } f(x)$$

The domain of $f \circ g$ is $[1, \infty)$. So, $(f \circ g)(2) = \sqrt{2 - 1} = 1$ is defined, but $(f \circ g)(0)$ is not defined because 0 is not in the domain of $f \circ g$.

CHECKPOINT Now try Exercise 35.

The composition of f with g is generally not the same as the composition of g with f. This is illustrated in Example 6.

Exploration

Let $f(x) = x + 2$ and $g(x) = 4 - x^2$. Are the compositions $f \circ g$ and $g \circ f$ equal? You can use your graphing utility to answer this question by entering and graphing the following functions.

$$y_1 = (4 - x^2) + 2$$
$$y_2 = 4 - (x + 2)^2$$

What do you observe? Which function represents $f \circ g$ and which represents $g \circ f$?

Example 6 Compositions of Functions

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, evaluate (a) $(f \circ g)(x)$ and (b) $(g \circ f)(x)$ when $x = 0$, 1, 2, and 3.

Algebraic Solution

a. $(f \circ g)(x) = f(g(x))$ — Definition of $f \circ g$

$= f(4 - x^2)$ — Definition of $g(x)$

$= (4 - x^2) + 2$ — Definition of $f(x)$

$= -x^2 + 6$

$(f \circ g)(0) = -0^2 + 6 = 6$

$(f \circ g)(1) = -1^2 + 6 = 5$

$(f \circ g)(2) = -2^2 + 6 = 2$

$(f \circ g)(3) = -3^2 + 6 = -3$

b. $(g \circ f)(x) = g(f(x))$ — Definition of $g \circ f$

$= g(x + 2)$ — Definition of $f(x)$

$= 4 - (x + 2)^2$ — Definition of $g(x)$

$= 4 - (x^2 + 4x + 4)$

$= -x^2 - 4x$

$(g \circ f)(0) = -0^2 - 4(0) = 0$

$(g \circ f)(1) = -1^2 - 4(1) = -5$

$(g \circ f)(2) = -2^2 - 4(2) = -12$

$(g \circ f)(3) = -3^2 - 4(3) = -21$

Note that $f \circ g \neq g \circ f$.

CHECKPOINT Now try Exercise 37.

Numerical Solution

a. You can use the *table* feature of a graphing utility to evaluate $f \circ g$ when $x = 0$, 1, 2, and 3. Enter $y_1 = g(x)$ and $y_2 = f(g(x))$ in the *equation editor* (see Figure 1.60). Then set the table to *ask* mode to find the desired function values (see Figure 1.61). Finally, display the table, as shown in Figure 1.62.

b. You can evaluate $g \circ f$ when $x = 0$, 1, 2, and 3 by using a procedure similar to that of part (a). You should obtain the table shown in Figure 1.63.

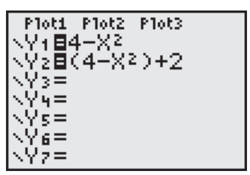


Figure 1.60

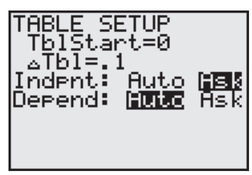


Figure 1.61

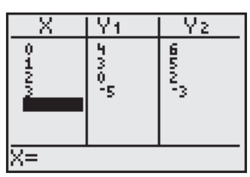


Figure 1.62

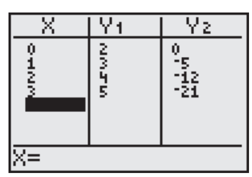


Figure 1.63

From the tables you can see that $f \circ g \neq g \circ f$.

To determine the domain of a composite function $f \circ g$, you need to restrict the outputs of g so that they are in the domain of f. For instance, to find the domain of $f \circ g$ given that $f(x) = 1/x$ and $g(x) = x + 1$, consider the outputs of g. These can be any real number. However, the domain of f is restricted to all real numbers except 0. So, the outputs of g must be restricted to all real numbers except 0. This means that $g(x) \neq 0$, or $x \neq -1$. So, the domain of $f \circ g$ is all real numbers except $x = -1$.

Example 7 Finding the Domain of a Composite Function

Find the domain of the composition $(f \circ g)(x)$ for the functions given by

$$f(x) = x^2 - 9 \qquad \text{and} \qquad g(x) = \sqrt{9 - x^2}.$$

Algebraic Solution

The composition of the functions is as follows.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\sqrt{9 - x^2}\right) \\ &= \left(\sqrt{9 - x^2}\right)^2 - 9 \\ &= 9 - x^2 - 9 \\ &= -x^2\end{aligned}$$

From this, it might appear that the domain of the composition is the set of all real numbers. This, however, is not true. Because the domain of f is the set of all real numbers and the domain of g is $[-3, 3]$, the domain of $(f \circ g)$ is $[-3, 3]$.

CHECKPOINT Now try Exercise 39.

Graphical Solution

You can use a graphing utility to graph the composition of the functions $(f \circ g)(x)$ as $y = \left(\sqrt{9 - x^2}\right)^2 - 9$. Enter the functions as follows.

$$y_1 = \sqrt{9 - x^2} \qquad y_2 = y_1^2 - 9$$

Graph y_2, as shown in Figure 1.64. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from -3 to 3. So, you can graphically estimate the domain of $(f \circ g)(x)$ to be $[-3, 3]$.

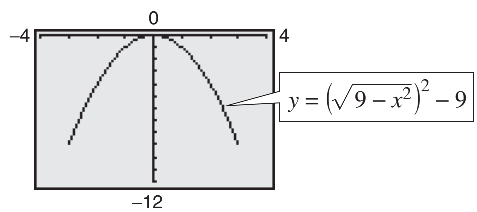


Figure 1.64

Example 8 A Case in Which $f \circ g = g \circ f$

Given $f(x) = 2x + 3$ and $g(x) = \frac{1}{2}(x - 3)$, find each composition.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$

Solution

a. $$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{2}(x - 3)\right) \\ &= 2\left[\frac{1}{2}(x - 3)\right] + 3 \\ &= x - 3 + 3 \\ &= x\end{aligned}$$

b. $$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x + 3) \\ &= \frac{1}{2}\left[(2x + 3) - 3\right] \\ &= \frac{1}{2}(2x) \\ &= x\end{aligned}$$

CHECKPOINT Now try Exercise 51.

STUDY TIP

In Example 8, note that the two composite functions $f \circ g$ and $g \circ f$ are equal, and both represent the identity function. That is, $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. You will study this special case in the next section.

In Examples 5, 6, 7, and 8, you formed the composition of two given functions. In calculus, it is also important to be able to identify two functions that make up a given composite function. Basically, to "decompose" a composite function, look for an "inner" and an "outer" function.

Example 9 Identifying a Composite Function

algebra of calculus

Write the function $h(x) = (3x - 5)^3$ as a composition of two functions.

Solution

One way to write h as a composition of two functions is to take the inner function to be $g(x) = 3x - 5$ and the outer function to be $f(x) = x^3$. Then you can write

$$h(x) = (3x - 5)^3$$
$$= f(3x - 5)$$
$$= f(g(x)).$$

CHECKPOINT Now try Exercise 65.

Example 10 Identifying a Composite Function

algebra of calculus

Write the function

$$h(x) = \frac{1}{(x-2)^2}$$

as a composition of two functions.

Solution

One way to write h as a composition of two functions is to take the inner function to be $g(x) = x - 2$ and the outer function to be

$$f(x) = \frac{1}{x^2}$$
$$= x^{-2}.$$

Then you can write

$$h(x) = \frac{1}{(x-2)^2}$$
$$= (x-2)^{-2}$$
$$= f(x-2)$$
$$= f(g(x)).$$

CHECKPOINT Now try Exercise 69.

Exploration

Write each function as a composition of two functions.

a. $h(x) = |x^3 - 2|$

b. $r(x) = |x^3| - 2$

What do you notice about the inner and outer functions?

Exploration

The function in Example 10 can be decomposed in other ways. For which of the following pairs of functions is $h(x)$ equal to $f(g(x))$?

a. $g(x) = \dfrac{1}{x-2}$ and $f(x) = x^2$

b. $g(x) = x^2$ and $f(x) = \dfrac{1}{x-2}$

c. $g(x) = \dfrac{1}{x}$ and $f(x) = (x-2)^2$

Example 11 Bacteria Count

The number N of bacteria in a refrigerated food is given by

$$N(T) = 20T^2 - 80T + 500, \qquad 2 \le T \le 14$$

where T is the temperature of the food (in degrees Celsius). When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 4t + 2, \qquad 0 \le t \le 3$$

where t is the time (in hours).

a. Find the composition $N(T(t))$ and interpret its meaning in context.

b. Find the number of bacteria in the food when $t = 2$ hours.

c. Find the time when the bacterial count reaches 2000.

Solution

a.
$$\begin{aligned} N(T(t)) &= 20(4t + 2)^2 - 80(4t + 2) + 500 \\ &= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\ &= 320t^2 + 320t + 80 - 320t - 160 + 500 \\ &= 320t^2 + 420 \end{aligned}$$

The composite function $N(T(t))$ represents the number of bacteria as a function of the amount of time the food has been out of refrigeration.

b. When $t = 2$, the number of bacteria is

$$\begin{aligned} N &= 320(2)^2 + 420 \\ &= 1280 + 420 \\ &= 1700. \end{aligned}$$

c. The bacterial count will reach $N = 2000$ when $320t^2 + 420 = 2000$. You can solve this equation for t algebraically as follows.

$$\begin{aligned} 320t^2 + 420 &= 2000 \\ 320t^2 &= 1580 \\ t^2 &= \frac{79}{16} \\ t &= \frac{\sqrt{79}}{4} \\ t &\approx 2.22 \text{ hours} \end{aligned}$$

So, the count will reach 2000 when $t \approx 2.22$ hours. When you solve this equation, note that the negative value is rejected because it is not in the domain of the composite function. You can use a graphing utility to confirm your solution. First graph the equation $N = 320t^2 + 420$, as shown in Figure 1.65. Then use the *zoom* and *trace* features to approximate $N = 2000$ when $t \approx 2.22$, as shown in Figure 1.66.

CHECKPOINT Now try Exercise 81.

Exploration

Use a graphing utility to graph $y_1 = 320x^2 + 420$ and $y_2 = 2000$ in the same viewing window. (Use a viewing window in which $0 \le x \le 3$ and $400 \le y \le 4000$.) Explain how the graphs can be used to answer the question asked in Example 11(c). Compare your answer with that given in part (c). When will the bacteria count reach 3200?
Notice that the model for this bacteria count situation is valid only for a span of 3 hours. Now suppose that the minimum number of bacteria in the food is reduced from 420 to 100. Will the number of bacteria still reach a level of 2000 within the three-hour time span? Will the number of bacteria reach a level of 3200 within 3 hours?

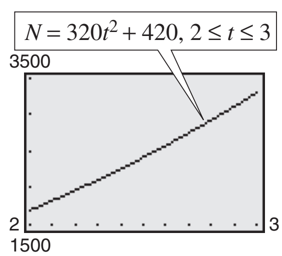


Figure 1.65

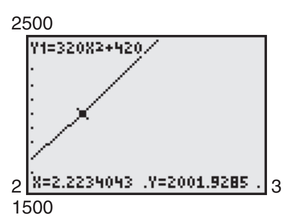


Figure 1.66

1.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

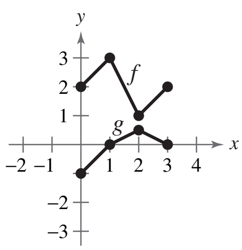
Vocabulary Check

Fill in the blanks.

1. Two functions f and g can be combined by the arithmetic operations of _______ , _______ , _______ , and _______ to create new functions.
2. The _______ of the function f with the function g is $(f \circ g)(x) = f(g(x))$.
3. The domain of $f \circ g$ is the set of all x in the domain of g such that _______ is in the domain of f.
4. To decompose a composite function, look for an _______ and an _______ function.

In Exercises 1–4, use the graphs of f and g to graph $h(x) = (f + g)(x)$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

1.



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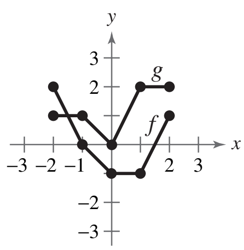

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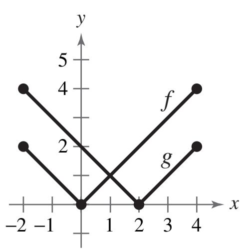

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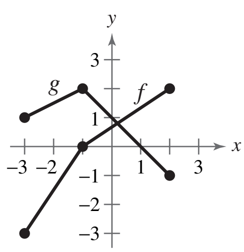


In Exercises 5–12, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?

5. $f(x) = x + 3, \quad g(x) = x - 3$
6. $f(x) = 2x - 5, \quad g(x) = 1 - x$
7. $f(x) = x^2, \quad g(x) = 1 - x$
8. $f(x) = 2x - 5, \quad g(x) = 4$
9. $f(x) = x^2 + 5, \quad g(x) = \sqrt{1 - x}$
10. $f(x) = \sqrt{x^2 - 4}, \quad g(x) = \dfrac{x^2}{x^2 + 1}$
11. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x^2}$
12. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = x^3$

In Exercises 13–26, evaluate the indicated function for $f(x) = x^2 - 1$ and $g(x) = x - 2$ algebraically. If possible, use a graphing utility to verify your answer.

13. $(f + g)(3)$
14. $(f - g)(-2)$
15. $(f - g)(0)$
16. $(f + g)(1)$
17. $(fg)(4)$
18. $(fg)(-6)$
19. $\left(\dfrac{f}{g}\right)(-5)$
20. $\left(\dfrac{f}{g}\right)(0)$
21. $(f - g)(2t)$
22. $(f + g)(t - 4)$
23. $(fg)(-5t)$
24. $(fg)(3t^2)$
25. $\left(\dfrac{f}{g}\right)(-t)$
26. $\left(\dfrac{f}{g}\right)(t + 2)$

In Exercises 27–30, use a graphing utility to graph the functions f, g, and h in the same viewing window.

27. $f(x) = \frac{1}{2}x, \quad g(x) = x - 1, \quad h(x) = f(x) + g(x)$
28. $f(x) = \frac{1}{3}x, \quad g(x) = -x + 4, \quad h(x) = f(x) - g(x)$
29. $f(x) = x^2, \quad g(x) = -2x, \quad h(x) = f(x) \cdot g(x)$
30. $f(x) = 4 - x^2, \quad g(x) = x, \quad h(x) = f(x)/g(x)$

In Exercises 31–34, use a graphing utility to graph f, g, and $f + g$ in the same viewing window. Which function contributes most to the magnitude of the sum when $0 \le x \le 2$? Which function contributes most to the magnitude of the sum when $x > 6$?

31. $f(x) = 3x, \quad g(x) = -\dfrac{x^3}{10}$
32. $f(x) = \dfrac{x}{2}, \quad g(x) = \sqrt{x}$
33. $f(x) = 3x + 2, \quad g(x) = -\sqrt{x + 5}$
34. $f(x) = x^2 - \frac{1}{2}, \quad g(x) = -3x^2 - 1$

In Exercises 35–38, find (a) $f \circ g$, (b) $g \circ f$, and, if possible, (c) $(f \circ g)(0)$.

35. $f(x) = x^2, \quad g(x) = x - 1$

36. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x^3 + 1$

37. $f(x) = 3x + 5, \quad g(x) = 5 - x$

38. $f(x) = x^3, \quad g(x) = \dfrac{1}{x}$

In Exercises 39–48, determine the domains of (a) f, (b) g, and (c) $f \circ g$. Use a graphing utility to verify your results.

39. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$

40. $f(x) = \sqrt{x + 3}, \quad g(x) = \dfrac{x}{2}$

41. $f(x) = x^2 + 1, \quad g(x) = \sqrt{x}$

42. $f(x) = x^{1/4}, \quad g(x) = x^4$

43. $f(x) = \dfrac{1}{x}, \quad g(x) = x + 3$

44. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{2x}$

45. $f(x) = |x - 4|, \quad g(x) = 3 - x$

46. $f(x) = \dfrac{2}{|x|}, \quad g(x) = x - 1$

47. $f(x) = x + 2, \quad g(x) = \dfrac{1}{x^2 - 4}$

48. $f(x) = \dfrac{3}{x^2 - 1}, \quad g(x) = x + 1$

In Exercises 49–54, (a) find $f \circ g$, $g \circ f$, and the domain of $f \circ g$. (b) Use a graphing utility to graph $f \circ g$ and $g \circ f$. Determine whether $f \circ g = g \circ f$.

49. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$

50. $f(x) = \sqrt[3]{x + 1}, \quad g(x) = x^3 - 1$

51. $f(x) = \frac{1}{3}x - 3, \quad g(x) = 3x + 9$

52. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x}$

53. $f(x) = x^{2/3}, \quad g(x) = x^6$

54. $f(x) = |x|, \quad g(x) = -x^2 + 1$

In Exercises 55–60, (a) find $(f \circ g)(x)$ and $(g \circ f)(x)$, (b) determine algebraically whether $(f \circ g)(x) = (g \circ f)(x)$, and (c) use a graphing utility to complete a table of values for the two compositions to confirm your answers to part (b).

55. $f(x) = 5x + 4, \quad g(x) = 4 - x$

56. $f(x) = \frac{1}{4}(x - 1), \quad g(x) = 4x + 1$

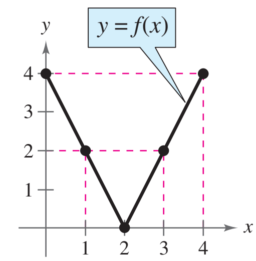
57. $f(x) = \sqrt{x + 6}, \quad g(x) = x^2 - 5$

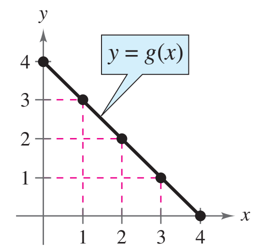
58. $f(x) = x^3 - 4, \quad g(x) = \sqrt[3]{x + 10}$

59. $f(x) = |x|, \quad g(x) = 2x^3$

60. $f(x) = \dfrac{6}{3x - 5}, \quad g(x) = -x$

In Exercises 61–64, use the graphs of f and g to evaluate the functions.





61. (a) $(f + g)(3)$ (b) $(f/g)(2)$

62. (a) $(f - g)(1)$ (b) $(fg)(4)$

63. (a) $(f \circ g)(2)$ (b) $(g \circ f)(2)$

64. (a) $(f \circ g)(1)$ (b) $(g \circ f)(3)$

In Exercises 65–72, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)

65. $h(x) = (2x + 1)^2$

66. $h(x) = (1 - x)^3$

67. $h(x) = \sqrt[3]{x^2 - 4}$

68. $h(x) = \sqrt{9 - x}$

69. $h(x) = \dfrac{1}{x + 2}$

70. $h(x) = \dfrac{4}{(5x + 2)^2}$

71. $h(x) = (x + 4)^2 + 2(x + 4)$

72. $h(x) = (x + 3)^{3/2} + 4(x + 3)^{1/2}$

73. ***Stopping Distance*** The research and development department of an automobile manufacturer has determined that when required to stop quickly to avoid an accident, the distance (in feet) a car travels during the driver's reaction time is given by

$$R(x) = \tfrac{3}{4}x$$

where x is the speed of the car in miles per hour. The distance (in feet) traveled while the driver is braking is given by

$$B(x) = \tfrac{1}{15}x^2.$$

(a) Find the function that represents the total stopping distance T.

(b) Use a graphing utility to graph the functions R, B, and T in the same viewing window for $0 \le x \le 60$.

(c) Which function contributes most to the magnitude of the sum at higher speeds? Explain.

74. ***Sales*** From 2000 to 2006, the sales R_1 (in thousands of dollars) for one of two restaurants owned by the same parent company can be modeled by $R_1 = 480 - 8t - 0.8t^2$, for $t = 0, 1, 2, 3, 4, 5, 6$, where $t = 0$ represents 2000. During the same seven-year period, the sales R_2 (in thousands of dollars) for the second restaurant can be modeled by $R_2 = 254 + 0.78t$, for $t = 0, 1, 2, 3, 4, 5, 6$.

(a) Write a function R_3 that represents the total sales for the two restaurants.

(b) Use a graphing utility to graph R_1, R_2, and R_3 (the total sales function) in the same viewing window.

Data Analysis **In Exercises 75 and 76, use the table, which shows the total amounts spent (in billions of dollars) on health services and supplies in the United States and Puerto Rico for the years 1995 through 2005. The variables y_1, y_2, and y_3 represent out-of-pocket payments, insurance premiums, and other types of payments, respectively.** (Source: U.S. Centers for Medicare and Medicaid Services)

Year	y_1	y_2	y_3
1995	146	330	457
1996	152	344	483
1997	162	361	503
1998	176	385	520
1999	185	414	550
2000	193	451	592
2001	202	497	655
2002	214	550	718
2003	231	601	766
2004	246	647	824
2005	262	691	891

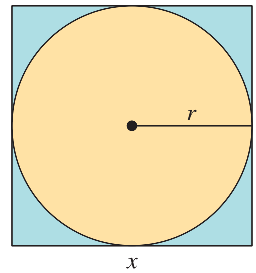
The models for this data are $y_1 = 11.4t + 83$, $y_2 = 2.31t^2 - 8.4t + 310$, and $y_3 = 3.03t^2 - 16.8t + 467$, where t represents the year, with $t = 5$ corresponding to 1995.

75. Use the models and the *table* feature of a graphing utility to create a table showing the values of y_1, y_2, and y_3 for each year from 1995 to 2005. Compare these values with the original data. Are the models a good fit? Explain.

76. Use a graphing utility to graph y_1, y_2, y_3, and $y_T = y_1 + y_2 + y_3$ in the same viewing window. What does the function y_T represent? Explain.

77. ***Ripples*** A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outermost ripple is given by $r(t) = 0.6t$, where t is the time (in seconds) after the pebble strikes the water. The area of the circle is given by $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

78. ***Geometry*** A square concrete foundation was prepared as a base for a large cylindrical gasoline tank (see figure).



(a) Write the radius r of the tank as a function of the length x of the sides of the square.

(b) Write the area A of the circular base of the tank as a function of the radius r.

(c) Find and interpret $(A \circ r)(x)$.

79. ***Cost*** The weekly cost C of producing x units in a manufacturing process is given by

$$C(x) = 60x + 750.$$

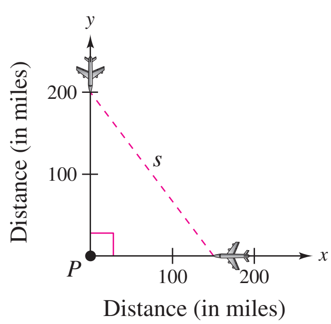
The number of units x produced in t hours is $x(t) = 50t$.

(a) Find and interpret $C(x(t))$.

(b) Find the number of units produced in 4 hours.

(c) Use a graphing utility to graph the cost as a function of time. Use the *trace* feature to estimate (to two-decimal-place accuracy) the time that must elapse until the cost increases to \$15,000.

80. ***Air Traffic Control*** An air traffic controller spots two planes at the same altitude flying toward each other. Their flight paths form a right angle at point P. One plane is 150 miles from point P and is moving at 450 miles per hour. The other plane is 200 miles from point P and is moving at 450 miles per hour. Write the distance s between the planes as a function of time t.



81. ***Bacteria*** The number of bacteria in a refrigerated food product is given by $N(T) = 10T^2 - 20T + 600$, for $1 \le T \le 20$, where T is the temperature of the food in degrees Celsius. When the food is removed from the refrigerator, the temperature of the food is given by $T(t) = 2t + 1$, where t is the time in hours.

(a) Find the composite function $N(T(t))$ or $(N \circ T)(t)$ and interpret its meaning in the context of the situation.

(b) Find $(N \circ T)(6)$ and interpret its meaning.

(c) Find the time when the bacteria count reaches 800.

82. ***Pollution*** The spread of a contaminant is increasing in a circular pattern on the surface of a lake. The radius of the contaminant can be modeled by $r(t) = 5.25\sqrt{t}$, where r is the radius in meters and t is time in hours since contamination.

(a) Find a function that gives the area A of the circular leak in terms of the time t since the spread began.

(b) Find the size of the contaminated area after 36 hours.

(c) Find when the size of the contaminated area is 6250 square meters.

83. ***Salary*** You are a sales representative for an automobile manufacturer. You are paid an annual salary plus a bonus of 3% of your sales over \$500,000. Consider the two functions

$f(x) = x - 500{,}000 \quad \text{and} \quad g(x) = 0.03x.$

If x is greater than \$500,000, which of the following represents your bonus? Explain.

(a) $f(g(x))$ (b) $g(f(x))$

84. ***Consumer Awareness*** The suggested retail price of a new car is p dollars. The dealership advertised a factory rebate of \$1200 and an 8% discount.

(a) Write a function R in terms of p giving the cost of the car after receiving the rebate from the factory.

(b) Write a function S in terms of p giving the cost of the car after receiving the dealership discount.

(c) Form the composite functions $(R \circ S)(p)$ and $(S \circ R)(p)$ and interpret each.

(d) Find $(R \circ S)(18{,}400)$ and $(S \circ R)(18{,}400)$. Which yields the lower cost for the car? Explain.

Synthesis

True or False? **In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.**

85. If $f(x) = x + 1$ and $g(x) = 6x$, then

$(f \circ g)(x) = (g \circ f)(x).$

86. If you are given two functions $f(x)$ and $g(x)$, you can calculate $(f \circ g)(x)$ if and only if the range of g is a subset of the domain of f.

Exploration **In Exercises 87 and 88, three siblings are of three different ages. The oldest is twice the age of the middle sibling, and the middle sibling is six years older than one-half the age of the youngest.**

87. (a) Write a composite function that gives the oldest sibling's age in terms of the youngest. Explain how you arrived at your answer.

(b) If the oldest sibling is 16 years old, find the ages of the other two siblings.

88. (a) Write a composite function that gives the youngest sibling's age in terms of the oldest. Explain how you arrived at your answer.

(b) If the youngest sibling is two years old, find the ages of the other two siblings.

89. ***Proof*** Prove that the product of two odd functions is an even function, and that the product of two even functions is an even function.

90. ***Conjecture*** Use examples to hypothesize whether the product of an odd function and an even function is even or odd. Then prove your hypothesis.

91. ***Proof*** Given a function f, prove that $g(x)$ is even and $h(x)$ is odd, where $g(x) = \frac{1}{2}[f(x) + f(-x)]$ and $h(x) = \frac{1}{2}[f(x) - f(-x)]$.

92. (a) Use the result of Exercise 91 to prove that any function can be written as a sum of even and odd functions. (*Hint:* Add the two equations in Exercise 91.)

(b) Use the result of part (a) to write each function as a sum of even and odd functions.

$$f(x) = x^2 - 2x + 1, \quad g(x) = \frac{1}{x+1}$$

Skills Review

In Exercises 93–96, find three points that lie on the graph of the equation. (There are many correct answers.)

93. $y = -x^2 + x - 5$

94. $y = \frac{1}{5}x^3 - 4x^2 + 1$

95. $x^2 + y^2 = 24$

96. $y = \dfrac{x}{x^2 - 5}$

In Exercises 97–100, find an equation of the line that passes through the two points.

97. $(-4, -2), (-3, 8)$

98. $(1, 5), (-8, 2)$

99. $\left(\frac{3}{2}, -1\right), \left(-\frac{1}{3}, 4\right)$

100. $(0, 1.1), (-4, 3.1)$

1.6 Inverse Functions

Inverse Functions

Recall from Section 1.2 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as follows.

$$f(x) = x + 4: \ \{(1, 5), (2, 6), (3, 7), (4, 8)\}$$

In this case, by interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of f, which is denoted by f^{-1}. It is a function from the set B to the set A, and can be written as follows.

$$f^{-1}(x) = x - 4: \ \{(5, 1), (6, 2), (7, 3), (8, 4)\}$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 1.67. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = f(x - 4) = (x - 4) + 4 = x$$

$$f^{-1}(f(x)) = f^{-1}(x + 4) = (x + 4) - 4 = x$$

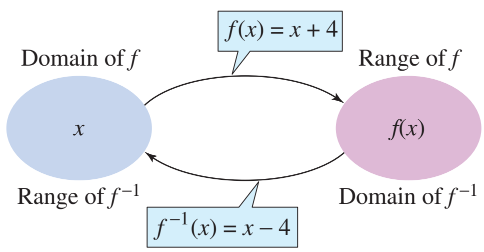


Figure 1.67

What you should learn

- Find inverse functions informally and verify that two functions are inverse functions of each other.
- Use graphs of functions to decide whether functions have inverse functions.
- Determine if functions are one-to-one.
- Find inverse functions algebraically.

Why you should learn it

Inverse functions can be helpful in further exploring how two variables relate to each other. For example, in Exercises 103 and 104 on page 71, you will use inverse functions to find the European shoe sizes from the corresponding U.S. shoe sizes.

LWA-Dann Tardif/Corbis

Example 1 Finding Inverse Functions Informally

Find the inverse function of $f(x) = 4x$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The function f *multiplies* each input by 4. To "undo" this function, you need to *divide* each input by 4. So, the inverse function of $f(x) = 4x$ is given by

$$f^{-1}(x) = \frac{x}{4}.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f\left(\frac{x}{4}\right) = 4\left(\frac{x}{4}\right) = x \qquad f^{-1}(f(x)) = f^{-1}(4x) = \frac{4x}{4} = x$$

CHECKPOINT Now try Exercise 1.

STUDY TIP

Don't be confused by the use of the exponent -1 to denote the inverse function f^{-1}. In this text, whenever f^{-1} is written, it always refers to the inverse function of the function f and not to the reciprocal of $f(x)$, which is given by

$$\frac{1}{f(x)}.$$

Example 2 Finding Inverse Functions Informally

Find the inverse function of $f(x) = x - 6$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The function f *subtracts* 6 from each input. To "undo" this function, you need to *add* 6 to each input. So, the inverse function of $f(x) = x - 6$ is given by

$$f^{-1}(x) = x + 6.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f(x + 6) = (x + 6) - 6 = x$$

$$f^{-1}(f(x)) = f^{-1}(x - 6) = (x - 6) + 6 = x$$

CHECKPOINT Now try Exercise 3.

A table of values can help you understand inverse functions. For instance, the following table shows several values of the function in Example 2. Interchange the rows of this table to obtain values of the inverse function.

x	-2	-1	0	1	2
$f(x)$	-8	-7	-6	-5	-4

x	-8	-7	-6	-5	-4
$f^{-1}(x)$	-2	-1	0	1	2

In the table at the left, each output is 6 less than the input, and in the table at the right, each output is 6 more than the input.

The formal definition of an inverse function is as follows.

Definition of Inverse Function

Let f and g be two functions such that

$f(g(x)) = x$ for every x in the domain of g

and

$g(f(x)) = x$ for every x in the domain of f.

Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

If the function g is the inverse function of the function f, it must also be true that the function f is the inverse function of the function g. For this reason, you can say that the functions f and g are *inverse functions of each other*.

Example 3 Verifying Inverse Functions Algebraically

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \qquad \text{and} \qquad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

Solution

$$\begin{aligned} f(g(x)) = f\left(\sqrt[3]{\frac{x+1}{2}}\right) &= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

$$\begin{aligned} g(f(x)) = g(2x^3 - 1) &= \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} \\ &= \sqrt[3]{\frac{2x^3}{2}} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned}$$

CHECKPOINT Now try Exercise 15.

Example 4 Verifying Inverse Functions Algebraically

Which of the functions is the inverse function of $f(x) = \dfrac{5}{x-2}$?

$$g(x) = \frac{x-2}{5} \qquad \text{or} \qquad h(x) = \frac{5}{x} + 2$$

Solution

By forming the composition of f with g, you have

$$f(g(x)) = f\left(\frac{x-2}{5}\right) = \frac{5}{\left(\dfrac{x-2}{5}\right) - 2} = \frac{25}{x - 12} \neq x.$$

Because this composition is not equal to the identity function x, it follows that g *is not* the inverse function of f. By forming the composition of f with h, you have

$$f(h(x)) = f\left(\frac{5}{x} + 2\right) = \frac{5}{\left(\dfrac{5}{x} + 2\right) - 2} = \frac{5}{\left(\dfrac{5}{x}\right)} = x.$$

So, it appears that h is the inverse function of f. You can confirm this by showing that the composition of h with f is also equal to the identity function.

CHECKPOINT Now try Exercise 19.

TECHNOLOGY TIP

Most graphing utilities can graph $y = x^{1/3}$ in two ways:

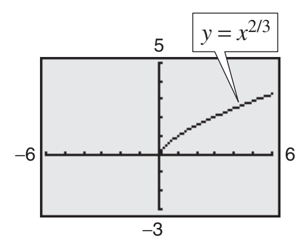
$y_1 = x \wedge (1/3)$ or

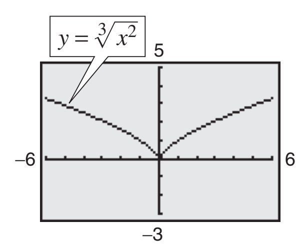
$y_1 = \sqrt[3]{x}$.

However, you may not be able to obtain the complete graph of $y = x^{2/3}$ by entering $y_1 = x \wedge (2/3)$. If not, you should use

$y_1 = (x \wedge (1/3))^2$ or

$y_1 = \sqrt[3]{x^2}$.





The Graph of an Inverse Function

The graphs of a function f and its inverse function f^{-1} are related to each other in the following way. If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. This means that the graph of f^{-1} is a *reflection* of the graph of f in the line $y = x$, as shown in Figure 1.68.

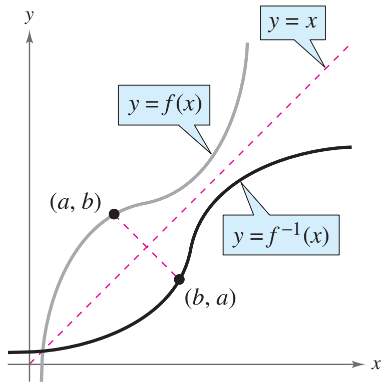


Figure 1.68

TECHNOLOGY TIP

In Examples 3 and 4, inverse functions were verified algebraically. A graphing utility can also be helpful in checking whether one function is the inverse function of another function. Use the Graph Reflection Program found at this textbook's *Online Study Center* to verify Example 4 graphically.

Example 5 Verifying Inverse Functions Graphically and Numerically

Verify that the functions f and g from Example 3 are inverse functions of each other graphically and numerically.

Graphical Solution

You can verify that f and g are inverse functions of each other *graphically* by using a graphing utility to graph f and g in the same viewing window. (Be sure to use a *square setting*.) From the graph in Figure 1.69, you can verify that the graph of g is the reflection of the graph of f in the line $y = x$.

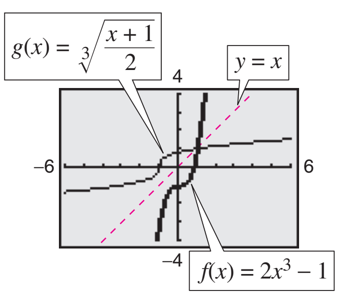


Figure 1.69

CHECKPOINT Now try Exercise 25.

Numerical Solution

You can verify that f and g are inverse functions of each other *numerically*. Begin by entering the compositions $f(g(x))$ and $g(f(x))$ into a graphing utility as follows.

$$y_1 = f(g(x)) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1$$

$$y_2 = g(f(x)) = \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}}$$

Then use the *table* feature of the graphing utility to create a table, as shown in Figure 1.70. Note that the entries for x, y_1, and y_2 are the same. So, $f(g(x)) = x$ and $g(f(x)) = x$. You can now conclude that f and g are inverse functions of each other.

X	Y1	Y2
-3	-3	-3
-2	-2	-2
-1	-1	-1
0	0	0
1	1	1
2	2	2
3	3	3

X=-3

Figure 1.70

The Existence of an Inverse Function

Consider the function $f(x) = x^2$. The first table at the right is a table of values for $f(x) = x^2$. The second table was created by interchanging the rows of the first table. The second table does not represent a function because the input $x = 4$ is matched with two different outputs: $y = -2$ and $y = 2$. So, $f(x) = x^2$ does not have an inverse function.

x	-2	-1	0	1	2
$f(x)$	4	1	0	1	4

x	4	1	0	1	4
$g(x)$	-2	-1	0	1	2

To have an inverse function, a function must be **one-to-one,** which means that no two elements in the domain of f correspond to the same element in the range of f.

Definition of a One-to-One Function

A function f is **one-to-one** if, for a and b in its domain, $f(a) = f(b)$ implies that $a = b$.

Existence of an Inverse Function

A function f has an inverse function f^{-1} if and only if f is one-to-one.

From its graph, it is easy to tell whether a function of x is one-to-one. Simply check to see that every horizontal line intersects the graph of the function at most once. This is called the **Horizontal Line Test.** For instance, Figure 1.71 shows the graph of $y = x^2$. On the graph, you can find a horizontal line that intersects the graph twice.

Two special types of functions that pass the Horizontal Line Test are those that are increasing or decreasing on their entire domains.

1. If f is *increasing* on its entire domain, then f is one-to-one.
2. If f is *decreasing* on its entire domain, then f is one-to-one.

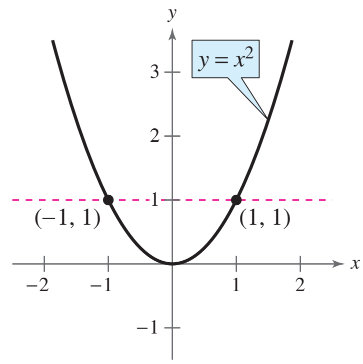


Figure 1.71 *$f(x) = x^2$ is not one-to-one.*

Example 6 Testing for One-to-One Functions

Is the function $f(x) = \sqrt{x} + 1$ one-to-one?

Algebraic Solution

Let a and b be nonnegative real numbers with $f(a) = f(b)$.

$$\sqrt{a} + 1 = \sqrt{b} + 1 \qquad \text{Set } f(a) = f(b).$$

$$\sqrt{a} = \sqrt{b}$$

$$a = b$$

So, $f(a) = f(b)$ implies that $a = b$. You can conclude that f *is* one-to-one and *does* have an inverse function.

CHECKPOINT Now try Exercise 55.

Graphical Solution

Use a graphing utility to graph the function $y = \sqrt{x} + 1$. From Figure 1.72, you can see that a horizontal line will intersect the graph at most once and the function is increasing. So, f *is* one-to-one and *does* have an inverse function.

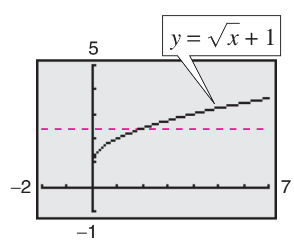


Figure 1.72

Finding Inverse Functions Algebraically

For simple functions, you can find inverse functions by inspection. For more complicated functions, however, it is best to use the following guidelines.

Finding an Inverse Function

1. Use the Horizontal Line Test to decide whether f has an inverse function.
2. In the equation for $f(x)$, replace $f(x)$ by y.
3. Interchange the roles of x and y, and solve for y.
4. Replace y by $f^{-1}(x)$ in the new equation.
5. Verify that f and f^{-1} are inverse functions of each other by showing that the domain of f is equal to the range of f^{-1}, the range of f is equal to the domain of f^{-1}, and $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

The function f with an implied domain of all real numbers may not pass the Horizontal Line Test. In this case, the domain of f may be restricted so that f does have an inverse function. For instance, if the domain of $f(x) = x^2$ is restricted to the nonnegative real numbers, then f does have an inverse function.

Example 7 Finding an Inverse Function Algebraically

Find the inverse function of $f(x) = \dfrac{5 - 3x}{2}$.

Solution

The graph of f in Figure 1.73 passes the Horizontal Line Test. So you know that f is one-to-one and has an inverse function.

$$f(x) = \frac{5 - 3x}{2} \qquad \text{Write original function.}$$

$$y = \frac{5 - 3x}{2} \qquad \text{Replace } f(x) \text{ by } y.$$

$$x = \frac{5 - 3y}{2} \qquad \text{Interchange } x \text{ and } y.$$

$$2x = 5 - 3y \qquad \text{Multiply each side by 2.}$$

$$3y = 5 - 2x \qquad \text{Isolate the } y\text{-term.}$$

$$y = \frac{5 - 2x}{3} \qquad \text{Solve for } y.$$

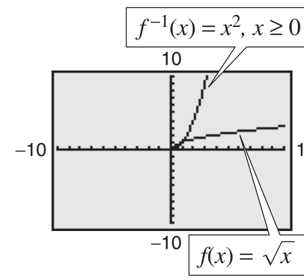
$$f^{-1}(x) = \frac{5 - 2x}{3} \qquad \text{Replace } y \text{ by } f^{-1}(x).$$

The domains and ranges of f and f^{-1} consist of all real numbers. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

CHECKPOINT Now try Exercise 59.

TECHNOLOGY TIP

Many graphing utilities have a built-in feature for drawing an inverse function. To see how this works, consider the function $f(x) = \sqrt{x}$. The inverse function of f is given by $f^{-1}(x) = x^2$, $x \geq 0$. Enter the function $y_1 = \sqrt{x}$. Then graph it in the standard viewing window and use the *draw inverse* feature. You should obtain the figure below, which shows both f and its inverse function f^{-1}. For instructions on how to use the *draw inverse* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.



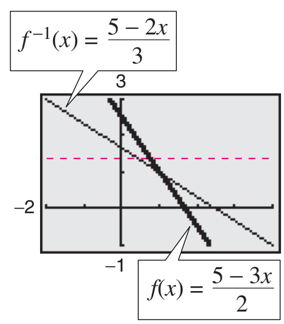


Figure 1.73

Example 8 Finding an Inverse Function Algebraically

Find the inverse function of $f(x) = x^3 - 4$ and use a graphing utility to graph f and f^{-1} in the same viewing window.

Solution

$f(x) = x^3 - 4$ Write original function.

$y = x^3 - 4$ Replace $f(x)$ by y.

$x = y^3 - 4$ Interchange x and y.

$y^3 = x + 4$ Isolate y.

$y = \sqrt[3]{x + 4}$ Solve for y.

$f^{-1}(x) = \sqrt[3]{x + 4}$ Replace y by $f^{-1}(x)$.

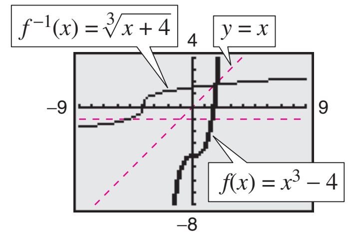


Figure 1.74

The graph of f in Figure 1.74 passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function. The graph of f^{-1} in Figure 1.74 is the reflection of the graph of f in the line $y = x$. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

CHECKPOINT Now try Exercise 61.

Example 9 Finding an Inverse Function Algebraically

Find the inverse function of $f(x) = \sqrt{2x - 3}$ and use a graphing utility to graph f and f^{-1} in the same viewing window.

Solution

$f(x) = \sqrt{2x - 3}$ Write original function.

$y = \sqrt{2x - 3}$ Replace $f(x)$ by y.

$x = \sqrt{2y - 3}$ Interchange x and y.

$x^2 = 2y - 3$ Square each side.

$2y = x^2 + 3$ Isolate y.

$y = \dfrac{x^2 + 3}{2}$ Solve for y.

$f^{-1}(x) = \dfrac{x^2 + 3}{2}, \quad x \geq 0$ Replace y by $f^{-1}(x)$.

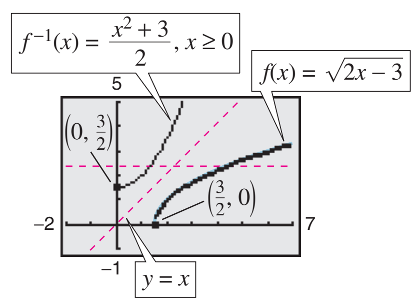


Figure 1.75

The graph of f in Figure 1.75 passes the Horizontal Line Test. So you know that f is one-to-one and has an inverse function. The graph of f^{-1} in Figure 1.75 is the reflection of the graph of f in the line $y = x$. Note that the range of f is the interval $[0, \infty)$, which implies that the domain of f^{-1} is the interval $[0, \infty)$. Moreover, the domain of f is the interval $\left[\frac{3}{2}, \infty\right)$, which implies that the range of f^{-1} is the interval $\left[\frac{3}{2}, \infty\right)$. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

CHECKPOINT Now try Exercise 65.

1.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. If the composite functions $f(g(x)) = x$ and $g(f(x)) = x$, then the function g is the _______ function of f, and is denoted by _______ .
2. The domain of f is the _______ of f^{-1}, and the _______ of f^{-1} is the range of f.
3. The graphs of f and f^{-1} are reflections of each other in the line _______ .
4. To have an inverse function, a function f must be _______ ; that is, $f(a) = f(b)$ implies $a = b$.
5. A graphical test for the existence of an inverse function is called the _______ Line Test.

In Exercises 1–8, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

1. $f(x) = 6x$

2. $f(x) = \frac{1}{3}x$

3. $f(x) = x + 7$

4. $f(x) = x - 3$

5. $f(x) = 2x + 1$

6. $f(x) = \dfrac{x - 1}{4}$

7. $f(x) = \sqrt[3]{x}$

8. $f(x) = x^5$

In Exercises 9–14, (a) show that f and g are inverse functions algebraically and (b) use a graphing utility to create a table of values for each function to numerically show that f and g are inverse functions.

9. $f(x) = -\dfrac{7}{2}x - 3, \quad g(x) = -\dfrac{2x + 6}{7}$

10. $f(x) = \dfrac{x - 9}{4}, \quad g(x) = 4x + 9$

11. $f(x) = x^3 + 5, \quad g(x) = \sqrt[3]{x - 5}$

12. $f(x) = \dfrac{x^3}{2}, \quad g(x) = \sqrt[3]{2x}$

13. $f(x) = -\sqrt{x - 8}, \quad g(x) = 8 + x^2, \quad x \le 0$

14. $f(x) = \sqrt[3]{3x - 10}, \quad g(x) = \dfrac{x^3 + 10}{3}$

In Exercises 15–20, show that f and g are inverse functions algebraically. Use a graphing utility to graph f and g in the same viewing window. Describe the relationship between the graphs.

15. $f(x) = x^3, \quad g(x) = \sqrt[3]{x}$

16. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$

17. $f(x) = \sqrt{x - 4}; \quad g(x) = x^2 + 4, \ x \ge 0$

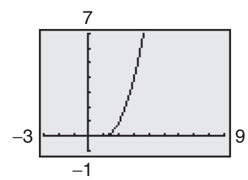
18. $f(x) = 9 - x^2, \quad x \ge 0; \quad g(x) = \sqrt{9 - x}$

19. $f(x) = 1 - x^3, \quad g(x) = \sqrt[3]{1 - x}$

20. $f(x) = \dfrac{1}{1 + x}, \ x \ge 0; \quad g(x) = \dfrac{1 - x}{x}, \ 0 < x \le 1$

In Exercises 21–24, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]

(a)



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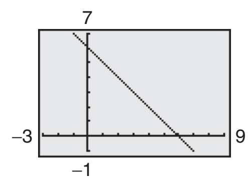
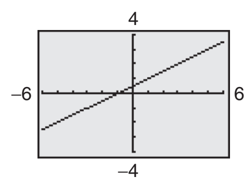


(c)



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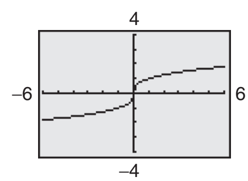

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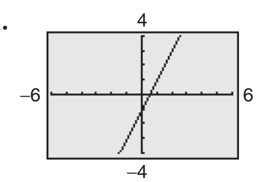

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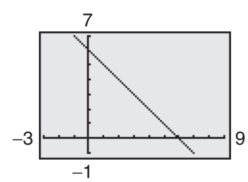

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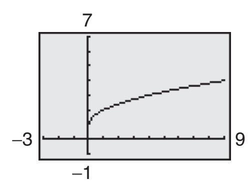

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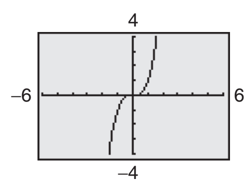


In Exercises 25–28, show that f and g are inverse functions (a) graphically and (b) numerically.

25. $f(x) = 2x, \quad g(x) = \dfrac{x}{2}$

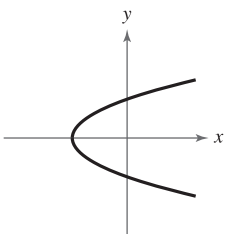
26. $f(x) = x - 5, \quad g(x) = x + 5$

27. $f(x) = \dfrac{x-1}{x+5}, \quad g(x) = -\dfrac{5x+1}{x-1}$

28. $f(x) = \dfrac{x+3}{x-2}, \quad g(x) = \dfrac{2x+3}{x-1}$

In Exercises 29–34, determine if the graph is that of a function. If so, determine if the function is one-to-one.

29.



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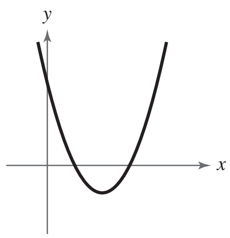

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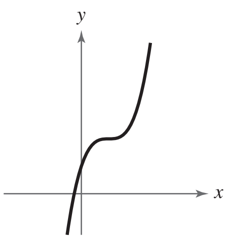

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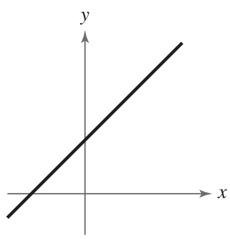

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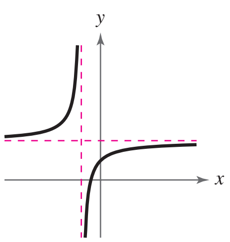

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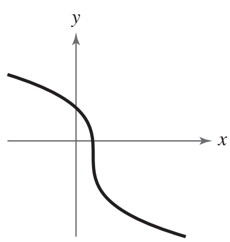


In Exercises 35–46, use a graphing utility to graph the function and use the Horizontal Line Test to determine whether the function is one-to-one and so has an inverse function exists.

35. $f(x) = 3 - \frac{1}{2}x$

36. $f(x) = \frac{1}{4}(x+2)^2 - 1$

37. $h(x) = \dfrac{x^2}{x^2+1}$

38. $g(x) = \dfrac{4-x}{6x^2}$

39. $h(x) = \sqrt{16 - x^2}$

40. $f(x) = -2x\sqrt{16 - x^2}$

41. $f(x) = 10$

42. $f(x) = -0.65$

43. $g(x) = (x+5)^3$

44. $f(x) = x^5 - 7$

45. $h(x) = |x+4| - |x-4|$

46. $f(x) = -\dfrac{|x-6|}{|x+6|}$

In Exercises 47–58, determine algebraically whether the function is one-to-one. Verify your answer graphically.

47. $f(x) = x^4$

48. $g(x) = x^2 - x^4$

49. $f(x) = \dfrac{3x+4}{5}$

50. $f(x) = 3x + 5$

51. $f(x) = \dfrac{1}{x^2}$

52. $h(x) = \dfrac{4}{x^2}$

53. $f(x) = (x+3)^2, \quad x \ge -3$

54. $q(x) = (x-5)^2, \quad x \le 5$

55. $f(x) = \sqrt{2x+3}$

56. $f(x) = \sqrt{x-2}$

57. $f(x) = |x-2|, \quad x \le 2$

58. $f(x) = \dfrac{x^2}{x^2+1}$

In Exercises 59–68, find the inverse function of f algebraically. Use a graphing utility to graph both f and f^{-1} in the same viewing window. Describe the relationship between the graphs.

59. $f(x) = 2x - 3$

60. $f(x) = 3x$

61. $f(x) = x^5$

62. $f(x) = x^3 + 1$

63. $f(x) = x^{3/5}$

64. $f(x) = x^2, \quad x \ge 0$

65. $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

66. $f(x) = \sqrt{16 - x^2}, \quad -4 \le x \le 0$

67. $f(x) = \dfrac{4}{x}$

68. $f(x) = \dfrac{6}{\sqrt{x}}$

Think About It **In Exercises 69–78, restrict the domain of the function f so that the function is one-to-one and has an inverse function. Then find the inverse function f^{-1}. State the domains and ranges of f and f^{-1}. Explain your results. (There are many correct answers.)**

69. $f(x) = (x-2)^2$

70. $f(x) = 1 - x^4$

71. $f(x) = |x+2|$

72. $f(x) = |x-2|$

73. $f(x) = (x+3)^2$

74. $f(x) = (x-4)^2$

75. $f(x) = -2x^2 + 5$

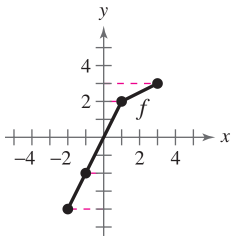
76. $f(x) = \frac{1}{2}x^2 - 1$

77. $f(x) = |x-4| + 1$

78. $f(x) = -|x-1| - 2$

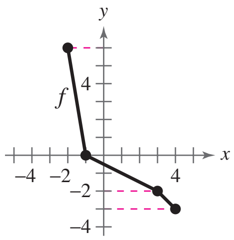
In Exercises 79 and 80, use the graph of the function f to complete the table and sketch the graph of f^{-1}.

79.



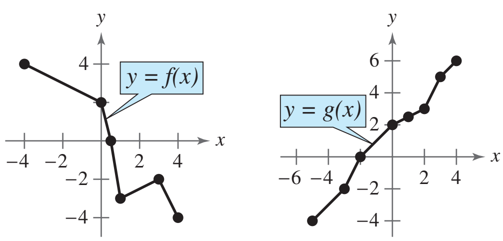
x	$f^{-1}(x)$
-4	
-2	
2	
3	

80.



x	$f^{-1}(x)$
-3	
-2	
0	
6	

In Exercises 81–88, use the graphs of $y = f(x)$ and $y = g(x)$ to evaluate the function.



81. $f^{-1}(0)$ **82.** $g^{-1}(0)$

83. $(f \circ g)(2)$ **84.** $g(f(-4))$

85. $f^{-1}(g(0))$ **86.** $(g^{-1} \circ f)(3)$

87. $(g \circ f^{-1})(2)$ **88.** $(f^{-1} \circ g^{-1})(-2)$

Graphical Reasoning **In Exercises 89–92, (a) use a graphing utility to graph the function, (b) use the *draw inverse* feature of the graphing utility to draw the inverse of the function, and (c) determine whether the graph of the inverse relation is an inverse function, explaining your reasoning.**

89. $f(x) = x^3 + x + 1$ **90.** $h(x) = x\sqrt{4 - x^2}$

91. $g(x) = \dfrac{3x^2}{x^2 + 1}$ **92.** $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

In Exercises 93–98, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the indicated value or function.

93. $(f^{-1} \circ g^{-1})(1)$ **94.** $(g^{-1} \circ f^{-1})(-3)$

95. $(f^{-1} \circ f^{-1})(6)$ **96.** $(g^{-1} \circ g^{-1})(-4)$

97. $(f \circ g)^{-1}$ **98.** $g^{-1} \circ f^{-1}$

In Exercises 99–102, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the specified function.

99. $g^{-1} \circ f^{-1}$ **100.** $f^{-1} \circ g^{-1}$

101. $(f \circ g)^{-1}$ **102.** $(g \circ f)^{-1}$

103. ***Shoe Sizes*** The table shows men's shoe sizes in the United States and the corresponding European shoe sizes. Let $y = f(x)$ represent the function that gives the men's European shoe size in terms of x, the men's U.S. size.

Men's U.S. shoe size	Men's European shoe size
8	41
9	42
10	43
11	45
12	46
13	47

(a) Is f one-to-one? Explain.

(b) Find $f(11)$.

(c) Find $f^{-1}(43)$, if possible.

(d) Find $f(f^{-1}(41))$.

(e) Find $f^{-1}(f(13))$.

104. ***Shoe Sizes*** The table shows women's shoe sizes in the United States and the corresponding European shoe sizes. Let $y = g(x)$ epresent the function that gives the women's European shoe size in terms of x, the women's U.S. size.

Women's U.S. shoe size	Women's European shoe size
4	35
5	37
6	38
7	39
8	40
9	42

(a) Is g one-to-one? Explain.

(b) Find $g(6)$.

(c) Find $g^{-1}(42)$.

(d) Find $g(g^{-1}(39))$.

(e) Find $g^{-1}(g(5))$.

105. ***Transportation*** The total values of new car sales f (in billions of dollars) in the United States from 1995 through 2004 are shown in the table. The time (in years) is given by t, with $t = 5$ corresponding to 1995. (Source: National Automobile Dealers Association)

Year, t	Sales, $f(t)$
5	456.2
6	490.0
7	507.5
8	546.3
9	606.5
10	650.3
11	690.4
12	679.5
13	699.2
14	714.3

(a) Does f^{-1} exist?

(b) If f^{-1} exists, what does it mean in the context of the problem?

(c) If f^{-1} exists, find $f^{-1}(650.3)$.

(d) If the table above were extended to 2005 and if the total value of new car sales for that year were \$690.4 billion, would f^{-1} exist? Explain.

106. ***Hourly Wage*** Your wage is \$8.00 per hour plus \$0.75 for each unit produced per hour. So, your hourly wage y in terms of the number of units produced x is $y = 8 + 0.75x$.

(a) Find the inverse function. What does each variable in the inverse function represent?

(b) Use a graphing utility to graph the function and its inverse function.

(c) Use the *trace* feature of a graphing utility to find the hourly wage when 10 units are produced per hour.

(d) Use the *trace* feature of a graphing utility to find the number of units produced when your hourly wage is \$22.25.

Synthesis

True or False? **In Exercises 107 and 108, determine whether the statement is true or false. Justify your answer.**

107. If f is an even function, f^{-1} exists.

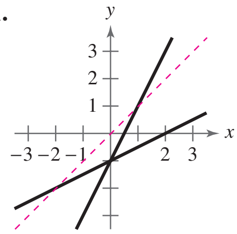
108. If the inverse function of f exists, and the graph of f has a y-intercept, the y-intercept of f is an x-intercept of f^{-1}.

109. ***Proof*** Prove that if f and g are one-to-one functions, $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

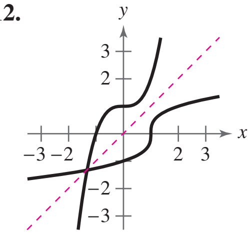
110. ***Proof*** Prove that if f is a one-to-one odd function, f^{-1} is an odd function.

In Exercises 111–114, decide whether the two functions shown in the graph appear to be inverse functions of each other. Explain your reasoning.

111.



112.



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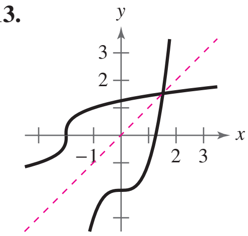
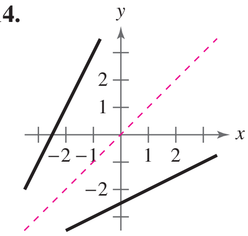


114.



In Exercises 115–118, determine if the situation could be represented by a one-to-one function. If so, write a statement that describes the inverse function.

115. The number of miles n a marathon runner has completed in terms of the time t in hours

116. The population p of South Carolina in terms of the year t from 1960 to 2005

117. The depth of the tide d at a beach in terms of the time t over a 24-hour period

118. The height h in inches of a human born in the year 2000 in terms of his or her age n in years

Skills Review

In Exercises 119–122, write the rational expression in simplest form.

119. $\dfrac{27x^3}{3x^2}$

120. $\dfrac{5x^2y}{xy + 5x}$

121. $\dfrac{x^2 - 36}{6 - x}$

122. $\dfrac{x^2 + 3x - 40}{x^2 - 3x - 10}$

In Exercises 123–128, determine whether the equation represents y as a function of x.

123. $4x - y = 3$

124. $x = 5$

125. $x^2 + y^2 = 9$

126. $x^2 + y = 8$

127. $y = \sqrt{x + 2}$

128. $x - y^2 = 0$

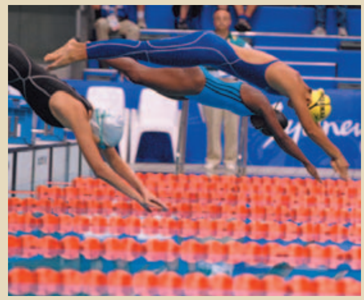
1.7 Linear Models and Scatter Plots

What you should learn

- Construct scatter plots and interpret correlation.
- Use scatter plots and a graphing utility to find linear models for data.

Why you should learn it

Real-life data often follows a linear pattern. For instance, in Exercise 20 on page 81, you will find a linear model for the winning times in the women's 400-meter freestyle Olympic swimming event.



Nick Wilson/Getty Images

Scatter Plots and Correlation

Many real-life situations involve finding relationships between two variables, such as the year and the outstanding household credit market debt. In a typical situation, data is collected and written as a set of ordered pairs. The graph of such a set is called a *scatter plot*. (For a brief discussion of scatter plots, see Appendix B.1.)

Example 1 Constructing a Scatter Plot

The data in the table shows the outstanding household credit market debt D (in trillions of dollars) from 1998 through 2004. Construct a scatter plot of the data. (Source: Board of Governors of the Federal Reserve System)

Year	Household credit market debt, D (in trillions of dollars)
1998	6.0
1999	6.4
2000	7.0
2001	7.6
2002	8.4
2003	9.2
2004	10.3

Solution

Begin by representing the data with a set of ordered pairs. Let t represent the year, with $t = 8$ corresponding to 1998.

$(8, 6.0), (9, 6.4), (10, 7.0), (11, 7.6), (12, 8.4), (13, 9.2), (14, 10.3)$

Then plot each point in a coordinate plane, as shown in Figure 1.76.

CHECKPOINT Now try Exercise 1.

Household Credit Market Debt

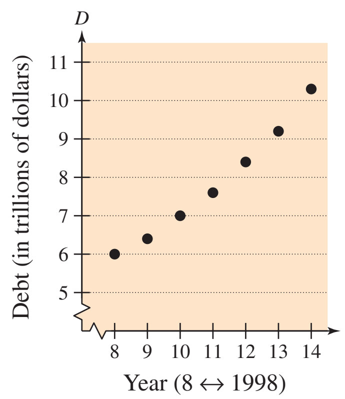


Figure 1.76

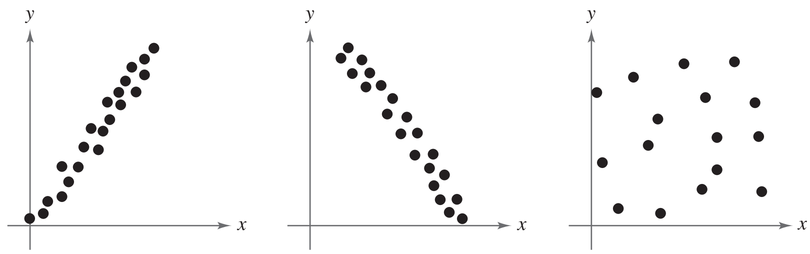
From the scatter plot in Figure 1.76, it appears that the points describe a relationship that is nearly linear. The relationship is not *exactly* linear because the household credit market debt did not increase by precisely the same amount each year.

A mathematical equation that approximates the relationship between t and D is a *mathematical model*. When developing a mathematical model to describe a set of data, you strive for two (often conflicting) goals—accuracy and simplicity. For the data above, a linear model of the form

$$D = at + b$$

appears to be best. It is simple and relatively accurate.

Consider a collection of ordered pairs of the form (x, y). If y tends to increase as x increases, the collection is said to have a **positive correlation.** If y tends to decrease as x increases, the collection is said to have a **negative correlation.** Figure 1.77 shows three examples: one with a positive correlation, one with a negative correlation, and one with no (discernible) correlation.



Positive Correlation ***Negative Correlation*** ***No Correlation***

Figure 1.77

Example 2 Interpreting Correlation

On a Friday, 22 students in a class were asked to record the numbers of hours they spent studying for a test on Monday and the numbers of hours they spent watching television. The results are shown below. (The first coordinate is the number of hours and the second coordinate is the score obtained on the test.)

Study Hours: (0, 40), (1, 41), (2, 51), (3, 58), (3, 49), (4, 48), (4, 64), (5, 55), (5, 69), (5, 58), (5, 75), (6, 68), (6, 63), (6, 93), (7, 84), (7, 67), (8, 90), (8, 76), (9, 95), (9, 72), (9, 85), (10, 98)

TV Hours: (0, 98), (1, 85), (2, 72), (2, 90), (3, 67), (3, 93), (3, 95), (4, 68), (4, 84), (5, 76), (7, 75), (7, 58), (9, 63), (9, 69), (11, 55), (12, 58), (14, 64), (16, 48), (17, 51), (18, 41), (19, 49), (20, 40)

a. Construct a scatter plot for each set of data.

b. Determine whether the points are positively correlated, are negatively correlated, or have no discernible correlation. What can you conclude?

Solution

a. Scatter plots for the two sets of data are shown in Figure 1.78.

b. The scatter plot relating study hours and test scores has a positive correlation. This means that the more a student studied, the higher his or her score tended to be. The scatter plot relating television hours and test scores has a negative correlation. This means that the more time a student spent watching television, the lower his or her score tended to be.

CHECKPOINT Now try Exercise 3.

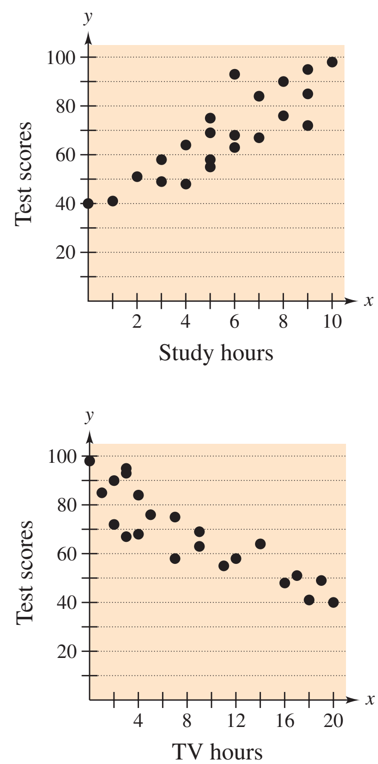


Figure 1.78

Fitting a Line to Data

Finding a linear model to represent the relationship described by a scatter plot is called **fitting a line to data.** You can do this graphically by simply sketching the line that appears to fit the points, finding two points on the line, and then finding the equation of the line that passes through the two points.

Example 3 Fitting a Line to Data

Find a linear model that relates the year to the outstanding household credit market debt. (See Example 1.)

Year	Household credit market debt, *D* (in trillions of dollars)
1998	6.0
1999	6.4
2000	7.0
2001	7.6
2002	8.4
2003	9.2
2004	10.3

Solution

Let t represent the year, with $t = 8$ corresponding to 1998. After plotting the data in the table, draw the line that you think best represents the data, as shown in Figure 1.79. Two points that lie on this line are (9, 6.4) and (13, 9.2). Using the point-slope form, you can find the equation of the line to be

$$D = 0.7(t - 9) + 6.4$$

$$= 0.7t + 0.1. \qquad \text{Linear model}$$

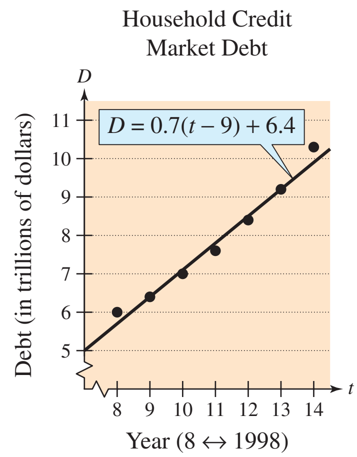


Figure 1.79

CHECKPOINT Now try Exercise 11(a) and (b).

Once you have found a model, you can measure how well the model fits the data by comparing the actual values with the values given by the model, as shown in the following table.

	t	8	9	10	11	12	13	14
Actual	D	6.0	6.4	7.0	7.6	8.4	9.2	10.3
Model	D	5.7	6.4	7.1	7.8	8.5	9.2	9.9

The sum of the squares of the differences between the actual values and the model values is the **sum of the squared differences.** The model that has the least sum is the **least squares regression line** for the data. For the model in Example 3, the sum of the squared differences is 0.31. The least squares regression line for the data is

$$D = 0.71t. \qquad \text{Best-fitting linear model}$$

Its sum of squared differences is 0.3015. See Appendix C for more on the least squares regression line.

STUDY TIP

The model in Example 3 is based on the two data points chosen. If different points are chosen, the model may change somewhat. For instance, if you choose (8, 6) and (14, 10.3), the new model is

$$D = 0.72(t - 8) + 6$$

$$= 0.72t + 0.24.$$

Example 4 A Mathematical Model

The numbers S (in billions) of shares listed on the New York Stock Exchange for the years 1995 through 2004 are shown in the table. (Source: New York Stock Exchange, Inc.)

Year	Shares, S
1995	154.7
1996	176.9
1997	207.1
1998	239.3
1999	280.9
2000	313.9
2001	341.5
2002	349.9
2003	359.7
2004	380.8

a. Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 5$ corresponding to 1995.

b. How closely does the model represent the data?

Graphical Solution

a. Enter the data into the graphing utility's list editor. Then use the *linear regression* feature to obtain the model shown in Figure 1.80. You can approximate the model to be $S = 26.47t + 29.0$.

b. You can use a graphing utility to graph the actual data and the model in the same viewing window. In Figure 1.81, it appears that the model is a fairly good fit for the actual data.

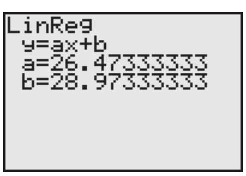


Figure 1.80

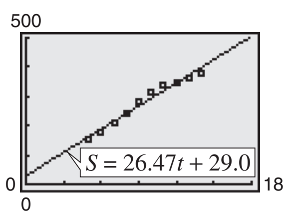


Figure 1.81

CHECKPOINT Now try Exercise 9.

Numerical Solution

a. Using the *linear regression* feature of a graphing utility, you can find that a linear model for the data is $S = 26.47t + 29.0$.

b. You can see how well the model fits the data by comparing the actual values of S with the values of S given by the model, which are labeled S^* in the table below. From the table, you can see that the model appears to be a good fit for the actual data.

Year	S	S^*
1995	154.7	161.4
1996	176.9	187.8
1997	207.1	214.3
1998	239.3	240.8
1999	280.9	267.2
2000	313.9	293.7
2001	341.5	320.2
2002	349.9	346.6
2003	359.7	373.1
2004	380.8	399.6

TECHNOLOGY SUPPORT

For instructions on how to use the *regression* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center.*

When you use the *regression* feature of a graphing calculator or computer program to find a linear model for data, you will notice that the program may also output an "r-value." For instance, the r-value from Example 4 was $r \approx 0.985$. This r-value is the **correlation coefficient** of the data and gives a measure of how well the model fits the data. Correlation coefficients vary between -1 and 1. Basically, the closer $|r|$ is to 1, the better the points can be described by a line. Three examples are shown in Figure 1.82.

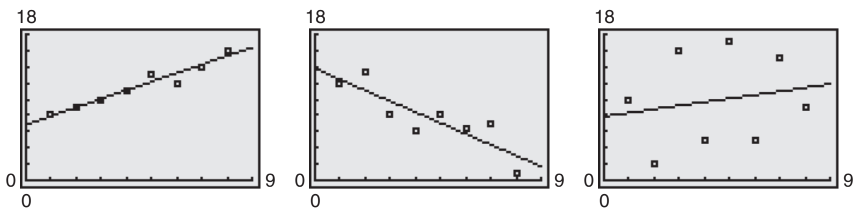


Figure 1.82

TECHNOLOGY TIP

For some calculators, the *diagnostics on* feature must be selected before the *regression* feature is used in order to see the r-value or correlation coefficient. To learn how to use this feature, consult your user's manual.

Example 5 Finding a Least Squares Regression Line

The following ordered pairs (w, h) represent the shoe sizes w and the heights h (in inches) of 25 men. Use the *regression* feature of a graphing utility to find the least squares regression line for the data.

(10.0, 70.5)	(10.5, 71.0)	(9.5, 69.0)	(11.0, 72.0)	(12.0, 74.0)
(8.5, 67.0)	(9.0, 68.5)	(13.0, 76.0)	(10.5, 71.5)	(10.5, 70.5)
(10.0, 71.0)	(9.5, 70.0)	(10.0, 71.0)	(10.5, 71.0)	(11.0, 71.5)
(12.0, 73.5)	(12.5, 75.0)	(11.0, 72.0)	(9.0, 68.0)	(10.0, 70.0)
(13.0, 75.5)	(10.5, 72.0)	(10.5, 71.0)	(11.0, 73.0)	(8.5, 67.5)

Solution

After entering the data into a graphing utility (see Figure 1.83), you obtain the model shown in Figure 1.84. So, the least squares regression line for the data is

$$h = 1.84w + 51.9.$$

In Figure 1.85, this line is plotted with the data. Note that the plot does not have 25 points because some of the ordered pairs graph as the same point. The correlation coefficient for this model is $r \approx 0.981$, which implies that the model is a good fit for the data.

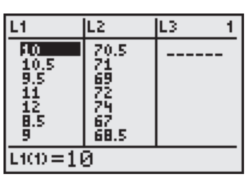


Figure 1.83

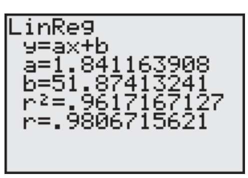


Figure 1.84

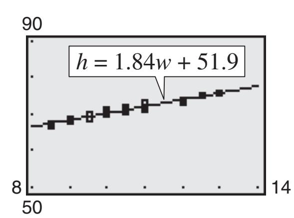


Figure 1.85

CHECKPOINT Now try Exercise 20.

1.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. Consider a collection of ordered pairs of the form (x, y). If y tends to increase as x increases, then the collection is said to have a _______ correlation.

2. Consider a collection of ordered pairs of the form (x, y). If y tends to decrease as x increases, then the collection is said to have a _______ correlation.

3. The process of finding a linear model for a set of data is called _______ .

4. Correlation coefficients vary between _______ and _______ .

---

1. ***Sales*** The following ordered pairs give the years of experience x for 15 sales representatives and the monthly sales y (in thousands of dollars).

(1.5, 41.7), (1.0, 32.4), (0.3, 19.2), (3.0, 48.4), (4.0, 51.2), (0.5, 28.5), (2.5, 50.4), (1.8, 35.5), (2.0, 36.0), (1.5, 40.0), (3.5, 50.3), (4.0, 55.2), (0.5, 29.1), (2.2, 43.2), (2.0, 41.6)

(a) Create a scatter plot of the data.

(b) Does the relationship between x and y appear to be approximately linear? Explain.

2. ***Quiz Scores*** The following ordered pairs give the scores on two consecutive 15-point quizzes for a class of 18 students.

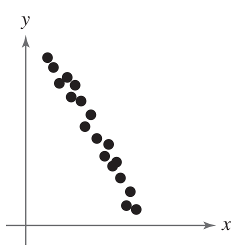
(7, 13), (9, 7), (14, 14), (15, 15), (10, 15), (9, 7), (14, 11), (14, 15), (8, 10), (9, 10), (15, 9), (10, 11), (11, 14), (7, 14), (11, 10), (14, 11), (10, 15), (9, 6)

(a) Create a scatter plot of the data.

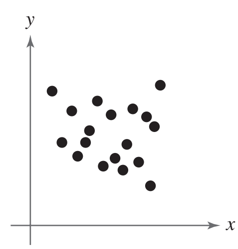
(b) Does the relationship between consecutive quiz scores appear to be approximately linear? If not, give some possible explanations.

In Exercises 3–6, the scatter plots of sets of data are shown. Determine whether there is positive correlation, negative correlation, or no discernible correlation between the variables.

3.



4.



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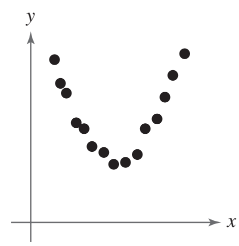

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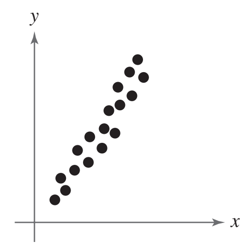
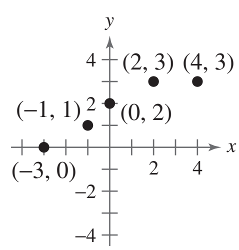


In Exercises 7–10, (a) for the data points given, draw a line of best fit through two of the points and find the equation of the line through the points, (b) use the *regression* feature of a graphing utility to find a linear model for the data, and to identify the correlation coefficient, (c) graph the data points and the lines obtained in parts (a) and (b) in the same viewing window, and (d) comment on the validity of both models. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

7.



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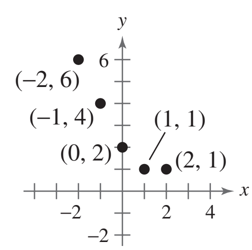
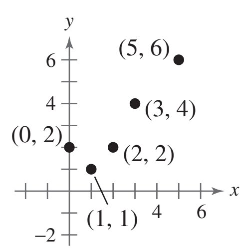
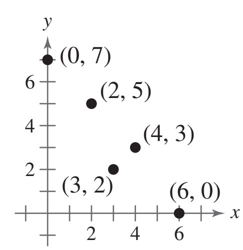


9.



10.



11. ***Hooke's Law*** Hooke's Law states that the force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is, $F = kd$, where k is the measure of the stiffness of the spring and is called the *spring constant*. The table shows the elongation d in centimeters of a spring when a force of F kilograms is applied.

Force, F	Elongation, d
20	1.4
40	2.5
60	4.0
80	5.3
100	6.6

(a) Sketch a scatter plot of the data.

(b) Find the equation of the line that seems to best fit the data.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data. Compare this model with the model from part (b).

(d) Use the model from part (c) to estimate the elongation of the spring when a force of 55 kilograms is applied.

12. ***Cell Phones*** The average lengths L of cellular phone calls in minutes from 1999 to 2004 are shown in the table. (Source: Cellular Telecommunications & Internet Association)

Year	Average length, L (in minutes)
1999	2.38
2000	2.56
2001	2.74
2002	2.73
2003	2.87
2004	3.05

(a) Use a graphing utility to create a scatter plot of the data, with $t = 9$ corresponding to 1999.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 9$ corresponding to 1999.

(c) Use a graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the average lengths of cellular phone calls for the years 2010 and 2015. Do your answers seem reasonable? Explain.

13. ***Sports*** The mean salaries S (in thousands of dollars) for professional football players in the United States from 2000 to 2004 are shown in the table. (Source: National Collegiate Athletic Assn.)

Year	Mean salary, S (in thousands of dollars)
2000	787
2001	986
2002	1180
2003	1259
2004	1331

(a) Use a graphing utility to create a scatter plot of the data, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(c) Use a graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the mean salaries for professional football players in 2005 and 2010. Do the results seem reasonable? Explain.

(e) What is the slope of your model? What does it tell you about the mean salaries of professional football players?

14. ***Teacher's Salaries*** The mean salaries S (in thousands of dollars) of public school teachers in the United States from 1999 to 2004 are shown in the table. (Source: Educational Research Service)

Year	Mean salary, S (in thousands of dollars)
1999	41.4
2000	42.2
2001	43.7
2002	43.8
2003	45.0
2004	45.6

(a) Use a graphing utility to create a scatter plot of the data, with $t = 9$ corresponding to 1999.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 9$ corresponding to 1999.

(c) Use a graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the mean salaries for teachers in 2005 and 2010. Do the results seem reasonable? Explain.

15. ***Cable Television*** The average monthly cable television bills C (in dollars) for a basic plan from 1990 to 2004 are shown in the table. (Source: Kagan Research, LLC)

Year	Monthly bill, C (in dollars)
1990	16.78
1991	18.10
1992	19.08
1993	19.39
1994	21.62
1995	23.07
1996	24.41
1997	26.48
1998	27.81
1999	28.92
2000	30.37
2001	32.87
2002	34.71
2003	36.59
2004	38.23

(a) Use a graphing utility to create a scatter plot of the data, with $t = 0$ corresponding to 1990.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the correlation coefficient. Let t represent the year, with $t = 0$ corresponding to 1990.

(c) Graph the model with the data in the same viewing window.

(d) Is the model a good fit for the data? Explain.

(e) Use the model to predict the average monthly cable bills for the years 2005 and 2010.

(f) Do you believe the model would be accurate to predict the average monthly cable bills for future years? Explain.

16. ***State Population*** The projected populations P (in thousands) for selected years for New Jersey based on the 2000 census are shown in the table. (Source: U.S. Census Bureau)

Year	Population, P (in thousands)
2005	8745
2010	9018
2015	9256
2020	9462
2025	9637
2030	9802

(a) Use a graphing utility to create a scatter plot of the data, with $t = 5$ corresponding to 2005.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 5$ corresponding to 2005.

(c) Use a graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the population of New Jersey in 2050. Does the result seem reasonable? Explain.

17. ***State Population*** The projected populations P (in thousands) for selected years for Wyoming based on the 2000 census are shown in the table. (Source: U.S. Census Bureau)

Year	Population, P (in thousands)
2005	507
2010	520
2015	528
2020	531
2025	529
2030	523

(a) Use a graphing utility to create a scatter plot of the data, with $t = 5$ corresponding to 2005.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 5$ corresponding to 2005.

(c) Use a graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the population of Wyoming in 2050. Does the result seem reasonable? Explain.

18. ***Advertising and Sales*** The table shows the advertising expenditures x and sales volumes y for a company for seven randomly selected months. Both are measured in thousands of dollars.

Month	Advertising expenditures, x	Sales volume, y
1	2.4	202
2	1.6	184
3	2.0	220
4	2.6	240
5	1.4	180
6	1.6	164
7	2.0	186

(a) Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the correlation coefficient.

(b) Use a graphing utility to plot the data and graph the model in the same viewing window.

(c) Interpret the slope of the model in the context of the problem.

(d) Use the model to estimate sales for advertising expenditures of \$1500.

19. ***Number of Stores*** The table shows the numbers T of Target stores from 1997 to 2006. (Source: Target Corp.)

Year	Number of stores, T
1997	1130
1998	1182
1999	1243
2000	1307
2001	1381
2002	1475
2003	1553
2004	1308
2005	1400
2006	1505

(a) Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the correlation coefficient. Let t represent the year, with $t = 7$ corresponding to 1997.

(b) Use a graphing utility to plot the data and graph the model in the same viewing window.

(c) Interpret the slope of the model in the context of the problem.

(d) Use the model to find the year in which the number of Target stores will exceed 1800.

(e) Create a table showing the actual values of T and the values of T given by the model. How closely does the model fit the data?

20. ***Sports*** The following ordered pairs (t, T) represent the Olympic year t and the winning time T (in minutes) in the women's 400-meter freestyle swimming event. (Source: *The World Almanac* 2005)

(1948, 5.30)	(1968, 4.53)	(1988, 4.06)
(1952, 5.20)	(1972, 4.32)	(1992, 4.12)
(1956, 4.91)	(1976, 4.16)	(1996, 4.12)
(1960, 4.84)	(1980, 4.15)	(2000, 4.10)
(1964, 4.72)	(1984, 4.12)	(2004, 4.09)

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 0$ corresponding to 1950.

(b) What information is given by the sign of the slope of the model?

(c) Use a graphing utility to plot the data and graph the model in the same viewing window.

(d) Create a table showing the actual values of y and the values of y given by the model. How closely does the model fit the data?

(e) Can the model be used to predict the winning times in the future? Explain.

Synthesis

True or False? **In Exercises 21 and 22, determine whether the statement is true or false. Justify your answer.**

21. A linear regression model with a positive correlation will have a slope that is greater than 0.

22. If the correlation coefficient for a linear regression model is close to -1, the regression line cannot be used to describe the data.

23. ***Writing*** A linear mathematical model for predicting prize winnings at a race is based on data for 3 years. Write a paragraph discussing the potential accuracy or inaccuracy of such a model.

24. ***Research Project*** Use your school's library, the Internet, or some other reference source to locate data that you think describes a linear relationship. Create a scatter plot of the data and find the least squares regression line that represents the points. Interpret the slope and y-intercept in the context of the data. Write a summary of your findings.

Skills Review

In Exercises 25–28, evaluate the function at each value of the independent variable and simplify.

25. $f(x) = 2x^2 - 3x + 5$

(a) $f(-1)$ (b) $f(w + 2)$

26. $g(x) = 5x^2 - 6x + 1$

(a) $g(-2)$ (b) $g(z - 2)$

27. $h(x) = \begin{cases} 1 - x^2, & x \le 0 \\ 2x + 3, & x > 0 \end{cases}$

(a) $h(1)$ (b) $h(0)$

28. $k(x) = \begin{cases} 5 - 2x, & x < -1 \\ x^2 + 4, & x \ge -1 \end{cases}$

(a) $k(-3)$ (b) $k(-1)$

In Exercises 29–34, solve the equation algebraically. Check your solution graphically.

29. $6x + 1 = -9x - 8$

30. $3(x - 3) = 7x + 2$

31. $8x^2 - 10x - 3 = 0$

32. $10x^2 - 23x - 5 = 0$

33. $2x^2 - 7x + 4 = 0$

34. $2x^2 - 8x + 5 = 0$

What Did You Learn?

Key Terms

slope, *p. 3*
point-slope form, *p. 5*
slope-intercept form, *p. 7*
parallel lines, *p. 9*
perpendicular lines, *p. 9*
function, *p. 16*
domain, *p. 16*
range, *p. 16*
independent variable, *p. 18*
dependent variable, *p. 18*
function notation, *p. 18*
graph of a function, *p. 30*
Vertical Line Test, *p. 31*
even function, *p. 36*
odd function, *p. 36*
rigid transformation, *p. 47*
inverse function, *p. 62*
one-to-one, *p. 66*
Horizontal Line Test, *p. 66*
positive correlation, *p. 74*
negative correlation, *p. 74*

Key Concepts

1.1 ■ Find and use the slopes of lines to write and graph linear equations

1. The slope m of the nonvertical line through (x_1, y_1) and (x_2, y_2), where $x_1 \neq x_2$, is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}.$$

2. The point-slope form of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is $y - y_1 = m(x - x_1)$.
3. The graph of the equation $y = mx + b$ is a line whose slope is m and whose y-intercept is $(0, b)$.

1.2 ■ Evaluate functions and find their domains

1. To evaluate a function $f(x)$, replace the independent variable x with a value and simplify the expression.
2. The domain of a function is the set of all real numbers for which the function is defined.

1.3 ■ Analyze graphs of functions

1. The graph of a function may have intervals over which the graph increases, decreases, or is constant.
2. The points at which a function changes its increasing, decreasing, or constant behavior are the relative minimum and relative maximum values of the function.
3. An even function is symmetric with respect to the y-axis. An odd function is symmetric with respect to the origin.

1.4 ■ Identify and graph shifts, reflections, and nonrigid transformations of functions

1. Vertical and horizontal shifts of a graph are transformations in which the graph is shifted up or down, and left or right.
2. A reflection transformation is a mirror image of a graph in a line.
3. A nonrigid transformation distorts the graph by stretching or shrinking the graph horizontally or vertically.

1.5 ■ Find arithmetic combinations and compositions of functions

1. An arithmetic combination of functions is the sum, difference, product, or quotient of two functions. The domain of the arithmetic combination is the set of all real numbers that are common to the two functions.
2. The composition of the function f with the function g is

   $(f \circ g)(x) = f(g(x))$.

   The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

1.6 ■ Find inverse functions

1. If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of its inverse function f^{-1}, and vice versa. This means that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$.
2. Use the Horizontal Line Test to decide if f has an inverse function. To find an inverse function algebraically, replace $f(x)$ by y, interchange the roles of x and y and solve for y, and replace y by $f^{-1}(x)$ in the new equation.

1.7 ■ Use scatter plots and find linear models

1. A scatter plot is a graphical representation of data written as a set of ordered pairs.
2. The best-fitting linear model can be found using the *linear regression* feature of a graphing utility or a computer program.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1.1 In Exercises 1 and 2, sketch the lines with the indicated slopes through the point on the same set of the coordinate axes.

Point	Slope	
1. $(1, 1)$	(a) 2	(b) 0
	(c) -1	(d) Undefined
2. $(-2, -3)$	(a) 1	(b) $-\frac{1}{2}$
	(c) 4	(d) 0

In Exercises 3–8, plot the two points and find the slope of the line passing through the points.

3. $(-3, 2), (8, 2)$

4. $(7, -1), (7, 12)$

5. $\left(\frac{3}{2}, 1\right), \left(5, \frac{5}{2}\right)$

6. $\left(-\frac{3}{4}, \frac{5}{6}\right), \left(\frac{1}{2}, -\frac{5}{2}\right)$

7. $(-4.5, 6), (2.1, 3)$

8 $(-2.7, -6.3), (-1, -1.2)$

In Exercises 9–18, (a) use the point on the line and the slope of the line to find the general form of the equation of the line, and (b) find three additional points through which the line passes. (There are many correct answers.)

Point	Slope
9. $(2, -1)$	$m = \frac{1}{4}$
10. $(-3, 5)$	$m = -\frac{3}{2}$
11. $(0, -5)$	$m = \frac{3}{2}$
12. $(3, 0)$	$m = -\frac{2}{3}$
13. $\left(\frac{1}{5}, -5\right)$	$m = -1$
14. $\left(0, \frac{7}{8}\right)$	$m = -\frac{4}{5}$
15. $(-2, 6)$	$m = 0$
16. $(-8, 8)$	$m = 0$
17. $(10, -6)$	m is undefined.
18. $(5, 4)$	m is undefined.

In Exercises 19–22, find the slope-intercept form of the equation of the line that passes through the points. Use a graphing utility to graph the line.

19. $(2, -1), (4, -1)$

20. $(0, 0), (0, 10)$

21. $(-1, 0), (6, 2)$

22. $(1, 6), (4, 2)$

Rate of Change **In Exercises 23–26, you are given the dollar value of a product in 2008 *and* the rate at which the value of the item is expected to change during the next 5 years. Use this information to write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 8$ represent 2008.)**

2008 Value	Rate
23. \$12,500	\$850 increase per year
24. \$3795	\$115 decrease per year
25. \$625.50	\$42.70 increase per year
26. \$72.95	\$5.15 decrease per year

27. ***Sales*** During the second and third quarters of the year, an e-commerce business had sales of \$160,000 and \$185,000, respectively. The growth of sales follows a linear pattern. Estimate sales during the fourth quarter.

28. ***Depreciation*** The dollar value of a DVD player in 2006 is \$225. The product will decrease in value at an expected rate of \$12.75 per year.

(a) Write a linear equation that gives the dollar value V of the DVD player in terms of the year t. (Let $t = 6$ represent 2006.)

(b) Use a graphing utility to graph the equation found in part (a). Be sure to choose an appropriate viewing window. State the dimensions of your viewing window, and explain why you chose the values that you did.

(c) Use the *value* or *trace* feature of your graphing utility to estimate the dollar value of the DVD player in 2010. Confirm your answer algebraically.

(d) According to the model, when will the DVD player have no value?

In Exercises 29–32, write the slope-intercept forms of the equations of the lines through the given point (a) parallel to the given line and (b) perpendicular to the given line. Verify your result with a graphing utility (use a *square setting*).

Point	Line
29. $(3, -2)$	$5x - 4y = 8$
30. $(-8, 3)$	$2x + 3y = 5$
31. $(-6, 2)$	$x = 4$
32. $(3, -4)$	$y = 2$

1.2 In Exercises 33 and 34, which sets of ordered pairs represent functions from A to B? Explain.

33. $A = \{10, 20, 30, 40\}$ and $B = \{0, 2, 4, 6\}$

(a) $\{(20, 4), (40, 0), (20, 6), (30, 2)\}$

(b) $\{(10, 4), (20, 4), (30, 4), (40, 4)\}$

(c) $\{(40, 0), (30, 2), (20, 4), (10, 6)\}$

(d) $\{(20, 2), (10, 0), (40, 4)\}$

34. $A = \{u, v, w\}$ and $B = \{-2, -1, 0, 1, 2\}$

(a) $\{(v, -1), (u, 2), (w, 0), (u, -2)\}$

(b) $\{(u, -2), (v, 2), (w, 1)\}$

(c) $\{(u, 2), (v, 2), (w, 1), (w, 1)\}$

(d) $\{(w, -2), (v, 0), (w, 2)\}$

In Exercises 35–38, determine whether the equation represents y as a function of x.

35. $16x^2 - y^2 = 0$

36. $2x - y - 3 = 0$

37. $y = \sqrt{1 - x}$

38. $|y| = x + 2$

In Exercises 39–42, evaluate the function at each specified value of the independent variable, and simplify.

39. $f(x) = x^2 + 1$

(a) $f(1)$ (b) $f(-3)$

(c) $f(b^3)$ (d) $f(x - 1)$

40. $g(x) = x^{4/3}$

(a) $g(8)$ (b) $g(t + 1)$

(c) $g(-27)$ (d) $g(-x)$

41. $h(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 + 2, & x > -1 \end{cases}$

(a) $h(-2)$ (b) $h(-1)$

(c) $h(0)$ (d) $h(2)$

42. $f(x) = \dfrac{3}{2x - 5}$

(a) $f(1)$ (b) $f(-2)$

(c) $f(t)$ (d) $f(10)$

In Exercises 43–48, find the domain of the function.

43. $f(x) = \dfrac{x - 1}{x + 2}$

44. $f(x) = \dfrac{x^2}{x^2 + 1}$

45. $f(x) = \sqrt{25 - x^2}$

46. $f(x) = \sqrt{x^2 - 16}$

47. $g(s) = \dfrac{5s + 5}{3s - 9}$ **48.** $f(x) = \dfrac{2x + 1}{3x + 4}$

49. ***Cost*** A hand tool manufacturer produces a product for which the variable cost is \$5.35 per unit and the fixed costs are \$16,000. The company sells the product for \$8.20 and can sell all that it produces.

(a) Write the total cost C as a function of x, the number of units produced.

(b) Write the profit P as a function of x.

50. ***Consumerism*** The retail sales R (in billions of dollars) of lawn care products and services in the United States from 1997 to 2004 can be approximated by the model

$$R(t) = \begin{cases} 0.126t^2 - 0.89t + 6.8, & 7 \le t < 11 \\ 0.1442t^3 - 5.611t^2 + 71.10t - 282.4, & 11 \le t \le 14 \end{cases}$$

where t represents the year, with $t = 7$ corresponding to 1997. Use the *table* feature of a graphing utility to approximate the retail sales of lawn care products and services for each year from 1997 to 2004. (Source: The National Gardening Association)

In Exercises 51 and 52, find the difference quotient and simplify your answer.

51. $f(x) = 2x^2 + 3x - 1, \quad \dfrac{f(x + h) - f(x)}{h}, \quad h \ne 0$

52. $f(x) = x^3 - 5x^2 + x, \quad \dfrac{f(x + h) - f(x)}{h}, \quad h \ne 0$

1.3 In Exercises 53–56, use a graphing utility to graph the function and estimate its domain and range. Then find the domain and range algebraically.

53. $f(x) = 3 - 2x^2$

54. $f(x) = \sqrt{2x^2 - 1}$

55. $h(x) = \sqrt{36 - x^2}$

56. $g(x) = |x + 5|$

In Exercises 57–60, (a) use a graphing utility to graph the equation and (b) use the Vertical Line Test to determine whether y is a function of x.

57. $y = \dfrac{x^2 + 3x}{6}$

58. $y = -\dfrac{2}{3}|x + 5|$

59. $3x + y^2 = 2$

60. $x^2 + y^2 = 49$

In Exercises 61–64, (a) use a graphing utility to graph the function and (b) determine the open intervals on which the function is increasing, decreasing, or constant.

61. $f(x) = x^3 - 3x$

62. $f(x) = \sqrt{x^2 - 9}$

63. $f(x) = x\sqrt{x - 6}$

64. $f(x) = \dfrac{|x + 8|}{2}$

In Exercises 65–68, use a graphing utility to approximate (to two decimal places) any relative minimum or relative maximum values of the function.

65. $f(x) = (x^2 - 4)^2$

66. $f(x) = x^2 - x - 1$

67. $h(x) = 4x^3 - x^4$

68. $f(x) = x^3 - 4x^2 - 1$

In Exercises 69–72, sketch the graph of the function by hand.

69. $f(x) = \begin{cases} 3x + 5, & x < 0 \\ x - 4, & x \geq 0 \end{cases}$

70. $f(x) = \begin{cases} x^2 + 7, & x < 1 \\ x^2 - 5x + 6, & x \geq 1 \end{cases}$

71. $f(x) = [\![x]\!] + 3$

72. $f(x) = [\![x + 2]\!]$

In Exercises 73–78, determine algebraically whether the function is even, odd, or neither. Verify your answer using a graphing utility.

73. $f(x) = x^2 + 6$

74. $f(x) = x^2 - x - 1$

75. $f(x) = (x^2 - 8)^2$

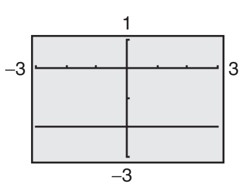
76. $f(x) = 2x^3 - x^2$

77. $f(x) = 3x^{5/2}$

78. $f(x) = 3x^{2/5}$

1.4 In Exercises 79–84, identify the parent function and describe the transformation shown in the graph. Write an equation for the graphed function.

79.



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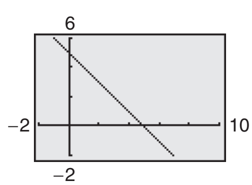

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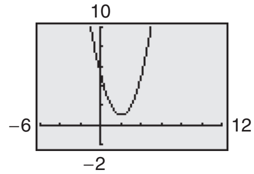

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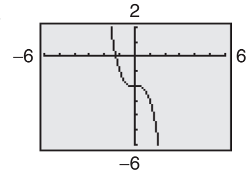

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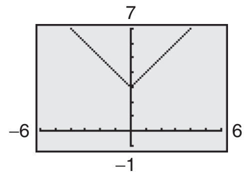

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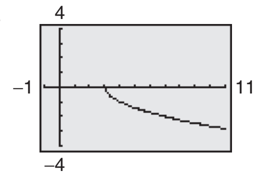
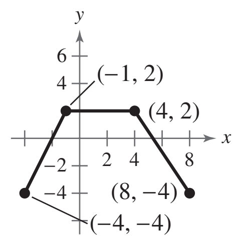


In Exercises 85–88, use the graph of $y = f(x)$ to graph the function.



85. $y = f(-x)$

86. $y = -f(x)$

87. $y = f(x) - 2$

88. $y = f(x - 1)$

Library of Parent Functions **In Exercises 89–100, h is related to one of the six parent functions on page 42. (a) Identify the parent function f. (b) Describe the sequence of transformations from f to h. (c) Sketch the graph of h by hand. (d) Use function notation to write h in terms of the parent function f.**

89. $h(x) = x^2 - 6$

90. $h(x) = -x^2 - 3$

91. $h(x) = (x - 2)^3 + 5$

92. $h(x) = -(x + 2)^2 - 8$

93. $h(x) = -(x - 2)^2 - 8$

94. $h(x) = \frac{1}{2}(x - 3)^2 - 6$

95. $h(x) = -\sqrt{x} + 5$

96. $h(x) = 2\sqrt{x} + 5$

97. $h(x) = \sqrt{x - 1} + 3$

98. $h(x) = |x| + 9$

99. $h(x) = -\frac{1}{2}|x| + 9$

100. $h(x) = |x + 8| - 1$

1.5 In Exercises 101–110, let $f(x) = 3 - 2x$, $g(x) = \sqrt{x}$, and $h(x) = 3x^2 + 2$, and find the indicated values.

101. $(f - g)(4)$

102. $(f + h)(5)$

103. $(f + g)(25)$

104. $(g - h)(1)$

105. $(fh)(1)$

106. $\left(\dfrac{g}{h}\right)(1)$

107. $(h \circ g)(7)$

108. $(g \circ f)(-2)$

109. $(f \circ h)(-4)$

110. $(g \circ h)(6)$

In Exercises 111–116, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)

111. $h(x) = (x + 3)^2$

112. $h(x) = (1 - 2x)^3$

113. $h(x) = \sqrt{4x + 2}$

114. $h(x) = \sqrt[3]{(x + 2)^2}$

115. $h(x) = \dfrac{4}{x + 2}$

116. $h(x) = \dfrac{6}{(3x + 1)^3}$

Data Analysis **In Exercises 117 and 118, the numbers (in millions) of students taking the SAT (y_1) and ACT (y_2) for the years 1990 through 2004 can be modeled by**

$$y_1 = 0.00204t^2 + 0.0015t + 1.021$$

and

$$y_2 = 0.0274t + 0.785$$

where t represents the year, with $t = 0$ corresponding to 1990. (Source: College Entrance Examination Board and ACT, Inc.)

117. Use a graphing utility to graph y_1, y_2, and $y_1 + y_2$ in the same viewing window.

118. Use the model $y_1 + y_2$ to estimate the total number of students taking the SAT and ACT in 2008.

1.6 In Exercises 119–122, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}f(x)) = x$.

119. $f(x) = 6x$

120. $f(x) = x + 5$

121. $f(x) = \dfrac{1}{2}x + 3$

122. $f(x) = \dfrac{x - 4}{5}$

In Exercises 123 and 124, show that f and g are inverse functions (a) graphically and (b) numerically.

123. $f(x) = 3 - 4x, \quad g(x) = \dfrac{3 - x}{4}$

124. $f(x) = \sqrt{x + 1}, \quad g(x) = x^2 - 1, x \geq 0$

In Exercises 125–128, use a graphing utility to graph the function and use the Horizontal Line Test to determine whether the function is one-to-one and an inverse function exists.

125. $f(x) = \frac{1}{2}x - 3$

126. $f(x) = (x - 1)^2$

127. $h(t) = \dfrac{2}{t - 3}$

128. $g(x) = \sqrt{x + 6}$

In Exercises 129–134, find the inverse function of f algebraically.

129. $f(x) = \frac{1}{2}x - 5$

130. $f(x) = \dfrac{7x + 3}{8}$

131. $f(x) = 4x^3 - 3$

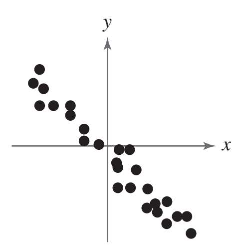
132. $f(x) = 5x^3 + 2$

133. $f(x) = \sqrt{x + 10}$

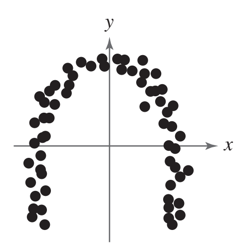
134. $f(x) = 4\sqrt{6 - x}$

1.7 In Exercises 135 and 136, the scatter plots of sets of data are shown. Determine whether there is positive correlation, negative correlation, or no discernible correlation between the variables.

135.



136.



137. ***Education*** The following ordered pairs give the entrance exam scores x and the grade-point averages y after 1 year of college for 10 students.

(75, 2.3), (82, 3.0), (90, 3.6), (65, 2.0), (70, 2.1),
(88, 3.5), (93, 3.9), (69, 2.0), (80, 2.8), (85, 3.3)

(a) Create a scatter plot of the data.

(b) Does the relationship between x and y appear to be approximately linear? Explain.

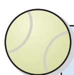
138. ***Stress Test*** A machine part was tested by bending it x centimeters 10 times per minute until it failed (y equals the time to failure in hours). The results are given as the following ordered pairs.

(3, 61), (6, 56), (9, 53), (12, 55), (15, 48), (18, 35),
(21, 36), (24, 33), (27, 44), (30, 23)

(a) Create a scatter plot of the data.

(b) Does the relationship between x and y appear to be approximately linear? If not, give some possible explanations.

139. ***Falling Object*** In an experiment, students measured the speed s (in meters per second) of a ball t seconds after it was released. The results are shown in the table.



Time, t	Speed, s
0	0
1	11.0
2	19.4
3	29.2
4	39.4

(a) Sketch a scatter plot of the data.

(b) Find the equation of the line that seems to fit the data best.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data and identify the correlation coefficient. Compare this model with the model from part (b).

(d) Use the model from part (c) to estimate the speed of the ball after 2.5 seconds.

140. ***Sports*** The following ordered pairs (x, y) represent the Olympic year x and the winning time y (in minutes) in the men's 400-meter freestyle swimming event. (Source: *The World Almanac* 2005)

(1964, 4.203)	(1980, 3.855)	(1996, 3.800)
(1968, 4.150)	(1984, 3.854)	(2000, 3.677)
(1972, 4.005)	(1988, 3.783)	(2004, 3.718)
(1976, 3.866)	(1992, 3.750)	

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let x represent the year, with $x = 4$ corresponding to 1964. Identify the correlation coefficient for the model.

(b) Use a graphing utility to create a scatter plot of the data.

(c) Graph the model with the data in the same viewing window.

(d) Does the model appear to be a good fit for the data? Explain.

(e) Would this model be appropriate for predicting the winning times in future Olympics? Explain.

Height **In Exercises 141–144, the following ordered pairs (x, y) represent the percent y of women between the ages of 20 and 29 who are under a certain height x (in feet).** (Source: U.S. National Center for Health Statistics)

(4.67, 0.6)	(5.17, 21.8)	(5.67, 92.4)
(4.75, 0.7)	(5.25, 34.3)	(5.75, 96.2)
(4.83, 1.2)	(5.33, 48.9)	(5.83, 98.6)
(4.92, 3.1)	(5.42, 62.7)	(5.92, 99.5)
(5.00, 6.0)	(5.50, 74.0)	(6.00, 100.0)
(5.08, 11.5)	(5.58, 84.7)	

141. Use the *regression* feature of a graphing utility to find a linear model for the data.

142. Use a graphing utility to plot the data and graph the model in the same viewing window.

143. How closely does the model fit the data?

144. Can the model be used to estimate the percent of women who are under a height of 6.3 feet?

Synthesis

True or False? **In Exercises 145–148, determine whether the statement is true or false. Justify your answer.**

145. If the graph of the parent function $f(x) = x^2$ is moved six units to the right, moved three units upward, and reflected in the x-axis, then the point $(-1, 28)$ will lie on the graph of the transformation.

146. If $f(x) = x^n$ where n is odd, f^{-1} exists.

147. There exists no function f such that $f = f^{-1}$.

148. The sign of the slope of a regression line is always positive.

1 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. After you are finished, check your work against the answers in the back of the book.

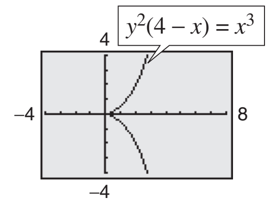


Figure for 3

1. Find the equations of the lines that pass through the point $(0, 4)$ and are (a) parallel to and (b) perpendicular to the line $5x + 2y = 3$.
2. Find the slope-intercept form of the equation of the line that passes through the points $(2, -1)$ and $(-3, 4)$.
3. Does the graph at the right represent y as a function of x? Explain.
4. Evaluate $f(x) = |x + 2| - 15$ at each value of the independent variable and simplify.

   (a) $f(-8)$ (b) $f(14)$ (c) $f(t - 6)$
5. Find the domain of $f(x) = 10 - \sqrt{3 - x}$.
6. An electronics company produces a car stereo for which the variable cost is \$5.60 and the fixed costs are \$24,000. The product sells for \$99.50. Write the total cost C as a function of the number of units produced and sold, x. Write the profit P as a function of the number of units produced and sold, x.

In Exercises 7 and 8, determine algebraically whether the function is even, odd, or neither.

7. $f(x) = 2x^3 - 3x$
8. $f(x) = 3x^4 + 5x^2$

In Exercises 9 and 10, determine the open intervals on which the function is increasing, decreasing, or constant.

9. $h(x) = \frac{1}{4}x^4 - 2x^2$
10. $g(t) = |t + 2| - |t - 2|$

In Exercises 11 and 12, use a graphing utility to graph the functions and to approximate (to two decimal places) any relative minimum or relative maximum values of the function.

11. $f(x) = -x^3 - 5x^2 + 12$
12. $f(x) = x^5 - x^3 + 2$

In Exercises 13–15, (a) identify the parent function f, (b) describe the sequence of transformations from f to g, and (c) sketch the graph of g.

13. $g(x) = -2(x - 5)^3 + 3$
14. $g(x) = \sqrt{-x - 7}$
15. $g(x) = 4|-x| - 7$
16. Use the functions $f(x) = x^2$ and $g(x) = \sqrt{2 - x}$ to find the specified function and its domain.

    (a) $(f - g)(x)$ (b) $\left(\frac{f}{g}\right)(x)$ (c) $(f \circ g)(x)$ (d) $(g \circ f)(x)$

In Exercises 17–19, determine whether the function has an inverse function, and if so, find the inverse function.

17. $f(x) = x^3 + 8$
18. $f(x) = x^2 + 6$
19. $f(x) = \dfrac{3x\sqrt{x}}{8}$
20. The table shows the numbers of cellular phone subscribers S (in millions) in the United States from 1999 through 2004, where t represents the year, with $t = 9$ corresponding to 1999. Use the *regression* feature of a graphing utility to find a linear model for the data. Use the model to find the year in which the number of subscribers exceeded 200 million. (Source: Cellular Telecommunications & Internet Association)

Year, t	Subscribers, S
9	86.0
10	109.5
11	128.4
12	140.8
13	158.7
14	182.1

Table for 20

Proofs in Mathematics

Conditional Statements

Many theorems are written in the **if-then form** "if p, then q," which is denoted by

$p \rightarrow q$ Conditional statement

where p is the **hypothesis** and q is the **conclusion.** Here are some other ways to express the conditional statement $p \rightarrow q$.

p implies q. p, only if q. p is sufficient for q.

Conditional statements can be either true or false. The conditional statement $p \rightarrow q$ is false only when p is true and q is false. To show that a conditional statement is true, you must prove that the conclusion follows for all cases that fulfill the hypothesis. To show that a conditional statement is false, you need only to describe a single **counterexample** that shows that the statement is not always true.

For instance, $x = -4$ is a counterexample that shows that the following statement is false.

If $x^2 = 16$, then $x = 4$.

The hypothesis "$x^2 = 16$" is true because $(-4)^2 = 16$. However, the conclusion "$x = 4$" is false. This implies that the given conditional statement is false.

For the conditional statement $p \rightarrow q$, there are three important associated conditional statements.

1. The **converse** of $p \rightarrow q$: $q \rightarrow p$
2. The **inverse** of $p \rightarrow q$: $\sim p \rightarrow \sim q$
3. The **contrapositive** of $p \rightarrow q$: $\sim q \rightarrow \sim p$

The symbol ~ means the **negation** of a statement. For instance, the negation of "The engine is running" is "The engine is not running."

Example 1 Writing the Converse, Inverse, and Contrapositive

Write the converse, inverse, and contrapositive of the conditional statement "If I get a B on my test, then I will pass the course."

Solution

a. *Converse:* If I pass the course, then I got a B on my test.

b. *Inverse:* If I do not get a B on my test, then I will not pass the course.

c. *Contrapositive:* If I do not pass the course, then I did not get a B on my test.

In the example above, notice that neither the converse nor the inverse is logically equivalent to the original conditional statement. On the other hand, the contrapositive *is* logically equivalent to the original conditional statement.

Biconditional Statements

Recall that a conditional statement is a statement of the form "if p, then q." A statement of the form "p if and only if q" is called a **biconditional statement.** A biconditional statement, denoted by

$p \leftrightarrow q$ Biconditional statement

is the conjunction of the conditional statement $p \rightarrow q$ and its converse $q \rightarrow p$.

A biconditional statement can be either true or false. To be true, *both* the conditional statement and its converse must be true.

Example 2 Analyzing a Biconditional Statement

Consider the statement $x = 3$ if and only if $x^2 = 9$.

a. Is the statement a biconditional statement? **b.** Is the statement true?

Solution

a. The statement is a biconditional statement because it is of the form "p if and only if q."

b. The statement can be rewritten as the following conditional statement and its converse.

Conditional statement: If $x = 3$, then $x^2 = 9$.
Converse: If $x^2 = 9$, then $x = 3$.

The first of these statements is true, but the second is false because x could also equal -3. So, the biconditional statement is false.

Knowing how to use biconditional statements is an important tool for reasoning in mathematics.

Example 3 Analyzing a Biconditional Statement

Determine whether the biconditional statement is true or false. If it is false, provide a counterexample.

A number is divisible by 5 if and only if it ends in 0.

Solution

The biconditional statement can be rewritten as the following conditional statement and its converse.

Conditional statement: If a number is divisible by 5, then it ends in 0.
Converse: If a number ends in 0, then it is divisible by 5.

The conditional statement is false. A counterexample is the number 15, which is divisible by 5 but does not end in 0. So, the biconditional statement is false.
